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GAUSS

〈1777—1855〉

SOME GREAT MATHEMATICIANS

OF THE

NINETEENTH CENTURY:

THEIR LIVES AND THEIR WORKS.

IN THREE VOLUMES.

VOL. I

WITH SIX PORTRAITS

(*Gauss, Cauchy, Abel, Jacobi, Weierstrass, Riemann*)

BY

GANESH PRASAD

HARDINGE PROFESSOR OF HIGHER MATHEMATICS IN THE CALCUTTA UNIVERSITY
LIFE PRESIDENT OF THE BENARES MATHEMATICAL SOCIETY, AND
PRESIDENT OF THE CALCUTTA MATHEMATICAL SOCIETY.

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1933.

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DEDICATED

TO THE MEMORIES

OF

MY REVERED PARENTS

RAMGOPAL SINGH

AND

JUTHAN DEBI

GANESH PRASAD

PREFACE.

The present volume, which is the first of the three volumes of my book and which deals with Gauss, Cauchy, Abel, Jacobi, Weierstrass and Riemann, six men who are universally accepted as the first six among the greatest mathematicians of the 19th century, is based on the six lectures which I delivered at the Calcutta University in January, February, March and April of 1932. In writing this volume, my aim has been to place before the minds of the readers the highest types of mathematicians, so that not only every young and accomplished mathematical scholar of to-day but also the budding mathematical genius of the future may strive to attain to the highest ideal of mathematical scholarship and research. Vol. 2, which deals with Cayley, Hermite, Brioschi, Poincaré, Klein and five others, is ready for the Press. Vol. 3 will be on some of the great living mathematicians.

In preparing the present volume, I have made use of a large number of writings of a historical and critical character, in addition to the collected works of the six mathematicians. Many of these writings have been quoted in this volume. But it gives me great pleasure to express here my special obligations to Klein's *Vorlesungen ueber die Entwicklung der Mathematik im 19. Jahrhundert*; Koeingsberger's Festschrift, *Gustav Jacob Jacobi*; Bjerknes's *Niels—Henrik Abel: Tableau de sa vie et de son action scientifique*; and the articles of Mittag-Leffler on Weierstrass in various volumes of the *Acta Mathematica*.

I take this opportunity to express my sincere thanks to a number of my friends and pupils who have helped me in various ways. The chapters dealing with Gauss and Cauchy were read by Dr. Bibhutibhuson Datta, D.Sc., the distinguished historian of the Hindu Mathematics of ancient and medieval times, both in manuscript and in proof, and his advice, given ungrudgingly in spite of the life of detachment from mundane affairs which he is leading, has materially improved the book. In connection with these chapters, I am also indebted to Mr. Hari Prasanna Banerjea, M.Sc, University Lecturer in Pure Mathematics at Calcutta. Mr. Chandi Prasad, M.A., B. Sc., Vice-Principal of Queen's College, Benares, has not only gone through many of the proof-sheets but has also been my representative with the Press at Benares during my stay at Calcutta. Father Albert Schelvis, S. J., of St. Xavier's College, Calcutta, has helped me with his scholarship in Latin whenever I needed his help. Mr Kuldip Sahai, M, Sc., one of my researchers at Calcutta, has helped me much by going through the proof-sheets of almost every chapter and by detecting many of the errata.

CALCUTTA. GANESH PRASAD

The 20th March, 1933.

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CHAPTER I.

GAUSS

Carl Friedrich Gauss was born at Brunswick on the 30th April, 1777. His father, Gebhard Dietrich Gauss (1744—1808), was a brick-layer in very humble circumstances, but an energetic and extremely conscientious man although strict and obstinate. Gauss's mother, to whom Gauss remained deeply attached all his life, was Dorothea, born Benze in 1742, who followed her son's rise in later life and died in 1839. She was born in a village five miles from Brunswick and came to Brunswick about the year 1769. She lived there for some years as a maid-servant before she was married to Gauss's father in 1776. She bore no other child than Gauss.

In a letter sent in 1810 to Minna Waldeck, who later married Gauss, Gauss gives many details* about his parents, grand-parents and step-brother. He says: "My grandfathers lived in the country and were, even if not altogether, still half, what one calls peasants. My paternal grand-father came to Brunswick about the year 1740, established himself there and had horticulture as the chief source of his livelihood." Of his father's first marriage, there was a son, Johann Georg Heinrich Gauss (1769—1854), who learnt tailoring, wandered away, returned home in 1794, became a soldier and then retired in 1806. After his father's death, Georg Gauss lived on horticulture, his income as a messenger and his income from his share of his father's property. In the letter, referred to above, Gauss says: "since 1½ years he has been married to a person about whom I do not know any thing further and has a small girl of ¼ year."

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* See Heinrich Mack's *C. F. Gauss und die Seinen*, E. Appelhans und Comp., Brunswick, 1927, pp. 70—73.

It was the wish of Gauss's father that Gauss should follow his own occupation. But the extraordinary capacity of the boy in the conception of numbers and the ease and surety with which he solved problems in mental Arithmetic were noticed at a very early age and filled with wonder all who came in touch with him.* Thus Gauss had to go in for higher education in spite of the opposition of his father.

Gauss received his earliest education at home and from 1784 to 1788 at a primary school of his native city. While a student at that school he met Johann

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* It is related by Sartorius v. Walterhausen that, when Gauss was hardly three years of age, he pointed out to his father a mistake in the distribution of the wages among the labourers working under him.

Martin Christian Bartels* (1769—1836), who was then an assistant teacher and later studied higher Mathematics and became Professor at the University of Dorpat. It is related† that at this school during Gauss's 10th year an event took place which produced a great impression on the teachers and the students. It was the custom that soon after a sum was set by the teacher the students would begin to work it and the first boy to finish the work would place his slate on the table, the others following in succession as soon as they became ready with the solution. It so happened that soon after his admission into the Arithmetic class, when a sum was set to the class Gauss put his slate within a minute of the announcement of the sum. After an hour many students finished the working and still it was found that Gauss's answer was right. It became then clear to the teacher, a man named Büttner, that Gauss had nothing to learn from him. Bartels felt such admiration for the talented boy of 10 years that he brought him to the notice of influential persons. Bartels also procured some mathematical books which he and Gauss studied together; and it was in this manner that, even before he had reached the age of 12 years, Gauss became acquainted* with the binomial theorem and the theory of infinite series.

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* He became later the father-in-law of the astronomer O. Struve. He lived under the protection of the Duke like Gauss at Brunswick about 1805; from 1807 to 1821 he taught at the University of Kasan and died as Professor at the University of Dorpat. The duty of Bartels in the school where Gauss studied was to make the quill-pens of the small boys and to help them in writing.

† See Sartorius von Walterhausen's *Gauss zum Gedächtniss*, Leipzig 1856, pp. 12—13

From 1788 to 1792, Gauss studied at the Gymnasium (or secondary school) where, because of his previous preparation, he was admitted into the second class, the highest being the first.† At the Gymnasium, Gauss's ability in classical languages was so distinguished as to be truly wonderful. His case came, through Geheimer Etatsrath von Zimmermann,‡ to the notice of the reigning Duke of Brunswick, Carl Wilhelm Ferdinand,§ who allowed Gauss to be introduced to him. From the age of 14 years up to the time of the death of this Prince in 1805, Gauss continued to enjoy his sympathy and pecuniary support.

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* In support of this remark there is Gauss's own statement in his letter to Olbers, dated the 6th April, 1802 (See Schering's *Gesammelte Mathematische Werke, Bd. 2, p, 193*)

† In each of the two highest classes, the course was of two years.

‡ Eberhard August Wilhelm von Zimmermann (1743–1815), Professor of Natural Sciences at the Collegium Carolinum.

§ Carl Wilhelm Ferdinand, nephew of the great Prussian General (and later Field-Marshal) Ferdinand, who was an intimate friend of Frederick the Great, was a great patron of learning. He had many fine qualities and had distinguished himself under his uncle in the Seven Years' War and had been described by Frederick the Great as 'ce héros dont l' esprit unite dés sa jeunesse le solide au brillant, l'ardeur á la sagesse'. But he was nearly 70 years old when he was appointed in 1806 Commander-in-chief of the Prussian forces against Napoleon. He was wanting in decision and held Napoleon in awe. The Duke was mortally wounded and defeated at Auerstadt. When a deputation from Brunswick waited on Napoleon, who was at the time at Halle, to beg him to let the Duke remain at

From 1792 to 1795, Gauss studied at the Collegium Carolinum of Brunswick-the seed from which the present Technical High School grew up-* and mastered the principal books of Newton, Lagrange and Euler, specially Newton's *Principia*. Whilst at that college, Gauss discovered for himself the famous theorem that the quadratic residue of all prime numbers of the form $4n+1$ is -1 and that -1 is the non-residue of all prime numbers of the form $4n+3$. This discovery was made, as the diary† of Gauss shows, in the March of 1795, and he found a proof of the theorem in the April of 1796. Euler who was the first discoverer—and Gauss came to know this later—had failed to obtain a proof even after repeated attempts extending over many years.

Gauss matriculated at the Göttingen University in the autumn of 1795 and continued to be a member

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Brunswick and die there in peace, Napoleon bluntly refused the petition with scorn. To prevent him from dying in prison, the Duke was taken in flight towards Altona where he died.

* See F. Klein's *Vorlesungen ueber die Entwicklung der Mathematik in 19. Jahrhundert*, Bd. I, Springer, Berlin, 1926, p. 24

† This diary or Notizenjournal (as Gauss calls it in a letter to Olbers) consists of 19 small octavo pages and was inherited by the family of Gauss, from whose representative, Herr C. Gauss of Hameln, the grandson of the great mathematician, it was borrowed by the Society of Sciences of Göttingen in 1898. The diary contains 146 items, the first bearing the date 30.3.1796 and the last the date 9.7.1814. The diary is reproduced in Gauss's *Werke*, Bd. X, erste Abtheilung.

of that University until the summer of 1798. According to E. Schering,* "it was during this brief period of the 19th, 20th and 21st years of his life, that this hero of Mathematics made his most ingenious discoveries and got glimpses of the seeds of a greater part of his later creations in different branches of scientific knowledge which belonged to the most difficult regions into which the power of the thought of man can penetrate." A month before this period of three years began, Gauss had made a discovery to which he always attached very great importance and which is supposed to have made him decide to devote all his life to Mathematics and the allied sciences. Also that discovery, which stands as the first item† in his diary, gave rise to his first publication which is in the shape of the following article‡ in the issue of the first June 1796, of the *Intelligenzblatt der Allgemeine Literatur-Zeitung* of Jena and Leipsic:—

"It is known to every beginner in Geometry that several ordinary polygons, viz, the triangle, square, fifteen-sided regular polygon and those which arise by repeatedly doubling the numbers of the sides of the above, admit of being constructed geometrically.

"So much was known in the time of Euclid and it appears that it has been generally assumed that the domain of Elementary Geometry cannot be extended further: at least I do not know of any successful attempt to extend the limits in this respect.

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* See Schering's *Gesammelte Math. Werke*, Bd: 2, p. 177.

† "Principia quibus innititur sectio circuli, ac divisibilitas eiusdem geometrica in septemdecim partes etc" Mart. 30. Brunsv.

‡ Reproduced in Gauss's *Werke*, Bd. X, p. 3; also Schering's *Werke l. c.*, p. 179.

"For this reason, it seems to me that the discovery is deserving of attention that, in addition to those ordinary polygons, a number of others, *e. g*, the 17-sided polygon, admit of being constructed geometrically. This result is only a corollary of a still incomplete theory of great extent and that theory will be presented to the public as soon as it is complete.

C. F. GAUSS AT BRUNSWICK,
student of Mathematics of Göttingen."

The stay of Gauss at the Göttingen University during his student days was fairly comfortable, as he was in receipt, from his Prince, of a stipend of 158 thalers* per year which sufficed for his needs; and, moreover, he was not at all anxious about his doctorate which he thought he could obtain whenever he liked. His chief preoccupation was the bringing out of his book on Higher Arithmetic (or Theory of Numbers) with the title *Disquisitiones Arithmeticae.* At that

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The article bears the date 18. April, '96 with the following note of Zimmermann: "It deserves to be noted that Hr. Gauss is at present in his 18th year, and has devoted himself here at Brunswick with the same fortunate success to Philosophy and to classical literature as to higher Mathematics."

1 thaler = 3 shillings in those days; also then in Germany 158 thalers was about one third of the salary of a privat-dozent (i.e., University lecturer).

time there was no book on the theory of numbers and it was the aim of Gauss to collect all his discoveries on the subject in the form of a comprehensive book, allotting to each of his predecessors in this field of investigation such credit as was due. For this purpose he consulted the University Library at Göttingen and made journeys to Helmstadt to use the University Library there. He made out carefully lists of mathematical journals and works, formed abstracts of papers and added his own critical observations on them. Roughly speaking, the first four out of the seven sections of the *Disquisitiones* contain mostly results in the discovery of which Gauss had been anticipated by previous writers like Fermat, Euler, Lagrange and Legendre. It is the last three sections which contain most of Gauss's own results. The book appears to have been sent to the press,* at least in part, in 1797, but its printing was delayed and it came out only in 1801.

At Göttingen, Gauss had, among his fellow students, Wolfgang von Bolyai,† in whom he found a congenial spirit and with whom he was very intimate. It is from the letters of Gauss to Bolyai sent from Brunswick or Helmstadt that we learn many of the difficulties and much of the working of his mind in the pecuniary situation in which he found himself after leaving Göttingen. Writing from Brunswick on the 29th November, 1798, Gauss says: "My situation is still very precarious and will probably continue to be so until my *Disquisitiones* is completed. I have not as yet spoken to the Duke. Von Zimmermann requested him in writing immediately after my arrival whether he would speak to me and to that no reply has come as yet. In case within a few days no reply comes or Zimmermann is able to discuss with him orally, which is likely to be the case, I shall make an attempt to go to him, although I feel almost certain that this will be purely an attempt, as no one is allowed to approach him unless he is called for. So necessary for me is his support—I am living at present chiefly on credit because all my financial prospects are shattered." Gauss was in those days in search of private tuition and, at the time of writing the letter from which the above quotation has been made, he had been hopeful of receiving help in that direction from Major—General von Stamford who was at Brunswick a short time before and had then been transferred to the Embassy at Berlin.

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* In November, 1798, Gauss complains to Bolyai as follows: "With my book, there is still very slow progress. I expect the proof-sheets of the 8th form to-day. The printer is a very phlegmatic man on whom all representations and requests make little impression." (see Schering's *Werke*, *l.c.*, p. 197). In a letter of 12.3. 1797 to Zimmermann, Gauss gives a plan of the book.

† Bolyai (1774—1855), the famous researcher in non-euclidean Geometry, studied at Göttingen where he met Gauss. Later Bolyai became Professor at the reformed college of Maros-Vasarhely in Transylvania.

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and after over 40 years of work there retired to his estate at Bolya in the neighbourhood of Maros-Vasarhely. See Schering's *Werke*, l. c., p. 197.

In January, 1799, Gauss could communicate to Bolyai the good news that the Prince had agreed to continue for the future the stipend which Gauss was receiving at Göttingen. The Prince also agreed to pay for the cost of the printing of Gauss's doctor-dissertation which was printed in August, 1799. The degree of Ph. D. had been conferred on him at Helmstadt* on the 16th July without his being troubled by many of the usual formalities.

The title of Gauss's-doctor dissertation is "Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primivel secundi gradus resolvi posse", and, in the words of Gauss,†"gives certainly the chief purpose of the dissertation, although this is brought out in about a third of the work only; the remaining portion contains principally the history and criticism of the works of other mathematicians on the subject, (namely, d' Alembert, Bougainville, Euler, de Foncenex, Lagrange and the writers of Compendia), together with remarks on the shallowness that prevails in our present-day Mathematics." Like other young men, Gauss wanted to know the opinions of his mathematical contemporaries on his dissertation and sent out 30 copies to various people, including Abraham Gotthelf Kästner (1714-1800) who was then the Professor of Mathematics at Göttingen.

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* The University of Helmstadt was founded by Julius of Brunswick in 1458.

† See Schering's *Werke, l. c*, p. 199, where the quotation occurs as taken from a letter to Bolyai, dated the 16th December, 1799.

After leaving Göttingen in the summer of 1798, Gauss spent most of his time at Brunswick, visiting Helmstadt now and then. There he had made the acquaintance of the Professor of Mathematics, Johann Friedrich Pfaff (1765—1825) earlier, and of him he spoke in the highest terms :* "I live here with Professor Pfaff whom I revere very much both as an excellent mathematician and as a good man and my warm friend, a man of innocent child-like character without the passions which dishonour men so much." Writing from Brunswick to Bolyai on the 3rd December, 1802, Gauss says†: "Since I left Helmstadt again in the Easter of 1800, I have been living here chiefly for my goddesses, the Sciences. Up to the summer, 1801, I was preoccupied with my great work, *Disquisitiones Arithmeticae* which came out‡ in Michaelmas, 1801.

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* See Schering's *Werke*, *l. c.*, pp. 200 and 201.

† See Schering's *Werke*, *l. c.*, pp. 201—202.

‡ The book was printed with the pecuniary support of the Duke and was not given to any regular publisher. But Gauss did not gain much from the sale, as many copies given to a Paris book-seller for sale on commission could not bring much money to Gauss on account of the bankruptcy of the book-seller. The book went out of print long before Gauss's death so that the well-known mathematician, Eisenstein (1823—1852), could not secure a copy. S. v. Walterhausen says in his biography of Gauss that many of the admirers of Gauss copied out with their own hands the whole book.

Since that time I have been engaged chiefly in connection with the new planets, Ceres and Pallas...... ....Our high-minded Prince has for the present, placed me in an independent care-free position by granting a pension of 400 thalers* a year. I am not very certain whether I shall enjoy this position long. Just now I am in correspondence on account of a call to St. Petersburgh as the Director of the Imperial Observatory and besides there is the chance of another call, so that I may have to choose between the two. However, I say this to you in closest confidence. The way in which my Fate decides will be known to you at once.

"Kästner's place is not yet filled up and will probably be not filled up at all. Mayer of Erlangen, who has come in the place of Lichtenberg, lectures on Mathematics also, and Thibaut, if I mistake not, has been made Extraordinary (i. e. Assistant) Professor. I am in very little touch with Göttingen. Only a few weeks ago, the Society of Sciences made me a corresponding member.

"Now may your good self live well. May the dream which we call life be to you sweeter, a foretaste of the real life in our proper home where the awakened soul is no more oppressed by the chains of the inert body, the limitations of space, the lash of earthly suffering and the vexations due to our petty wants and wishes. Let us carry the burden up to the end courageously and without murmur, but let us never lose sight of that higher ideal. Then when our hour strikes we will gladly let our burden drop and see the thick curtain fall."

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* equal to £ 60, the salary given to Dirichlet (1805–1859) as privat-dozent at Breslau in 1826.

Writing from Brunswick in June, 1803, Gauss says to Bolyai.* "The call to St. Petersburgh has not taken me away from here; our Duke does not let me go and has made my position here still more comfortable. I have indeed hope to get a small observatory here—if the war does not obstruct our plans. Astronomy and the Theory of Pure Numbers are the magnetic poles to which my mind's compass always turns."

Although the hopes of Gauss for an astronomical observatory at Brunswick were shattered by the defeat and death of his Prince in 1806, soon after that event he was appointed in 1807 at the Göttingen University as the Director of the Observatory† and Professor of

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* See Schering's *Werke*, *l. c*, pp. 202—203.

† Gauss received this call to Göttingen in the summer of 1807 at Bremen during his visit to Olbers (1758—1840), the famous astronomer, whom he asked for advice. Gauss also visited at Rehberg the Curator of the Göttingen University, Brande. After these consultations he entered on his duties in November, 1807. As the whole of the kingdom of Hanover was occupied by the French, the Hanoverian regime was in a state of dissolution; although some offices still existed and were in touch with the cabinet in London. In this confusion, no one asked Gauss to take the oath of allegiance to any one. Under Jerome Bonaparte, Göttingen became a part of the kingdom of Westphalia, and the building of the

Mathematics, a position which he occupied until his death about 48 years later. The year 1807 marked a turning point in the career of Gauss as a researcher and it will be appropriate for us to give an idea of his mode of work up to this year, making use, for this purpose, of the diary which Gauss kept from the year 1796 to 1814 and in which he noted down the discoveries made from time to time, although for the years after 1801 there are many gaps. (*a*) For the years before 1796, called by Klein* the "prehistoric period", it may be said that like his predecessors, Fermat and Euler, Gauss made discoveries in the Theory of Numbers by, what may be called, guess after working out special cases. In this work, his extraordinary patience and power of numerical calculation was of great use. He framed long tables† giving prime numbers, the quadratic residues and non-residues, the fractions $\frac{1}{p}$ in decimals for values of p from 1 to 1000 with full periods in order to detect the way in which the period depended on p. After guessing a theorem, he proceeded to give a rigorous proof of it. He was full of a kind of "child-like curiousity" and in this "experimental manner" came to learn general relations and theorems

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observatory was seriously expedited from 1811 onwards until the work of construction and equipment was finished in 1816 during the Hanoverian regime which had been re-established several years before.

* *l. c.*, p. 31.

† See Gauss's *Werke*, Bd. 2.

In the Theory of Numbers. In this period, Gauss discovered also the method of least squares. (*b*) From 1796 to 1801 was Gauss's second period of activity during which he entered in his diary almost every discovery. "Here we come across not the inaccessible, reserved and far-seeing man, but we see Gauss as he makes his great discoveries and becomes sensible of them. He gives expression to his joy and satisfaction in the most lively manner, and takes to himself flattering attributes and breaks out in exclamations. We see before us the proud array of the great discoveries in Arithmetic, Algebra and Analysis and the genesis of the *Disquisitiones Arithmeticae.* Almost touching appears to the reader, to find, among these traces of a violently breaking out genius, the signs of school work of elementary nature which could not be avoided even by a Gauss. Thus we find a list of exercises in differentiation, and immediately before the division of the lemniscate there are quite ordinary integral—substitutions which every student has to use."*

In spite of attempts† to draw him to Dorpat, Berlin or St. Petersburgh, Gauss continued to fill his post at

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* See Klein's *Entwicklung,* l. c., p. 33

† It was at one time contemplated by the Prussian Government to establish at Berlin an institution modelled after the Ecole Polytechnique of Paris and Gauss was to be the Director with no obligation to lecture. The new institution did not materialize. The offers to Dorpat and St. Petersburgh fell through because of the pension condition being not

Göttingen until his death, and was so devoted to his duties that, during the years 1827–1854, he never slept away from his observatory except once, when he accepted an invitation from Alexander von Humboldt (1769–1859) in 1828 to attend a scientific meeting at Berlin. But lecturing did not suit the inclinations of Gauss nor did the routine duties of a resident astronomer.* He could not build up a school round him of researchers in Mathematics in which subject he had few pupils, many of his pupils being in Astronomy. As Klein has said, Gauss belonged to the type to which Newton and Euler belonged–the type of the researcher who is international in his outlook and is concerned chiefly with his own researches. Almost all the publications of Gauss were in Latin and were intended chiefly for the learned world outside Germany which continued to send up to 1830 its brilliant sons like Dirichlet to Paris which remained up to 1850 the seat of the greatest school of Mathematics in Europe. But, although Gauss was not a great teacher in the ordinary sense, his position in the mathematical world excited feelings of the greatest reverence in the minds of his students. Writing in October, 1852, to his parents, the 19 years old student, E. Schering (1833–1897), says:† "I was never more fortunate than

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acceptable. It is doubtful if Gauss would have ever actually left the Göttingen University.

* See his letter to Bessel of 28.6.1820.

† See his *Werke, l. c*, p. 450.

on Monday last week when I heard that Gauss would probably deliver a course of lectures; the following night I slept very little. What feelings passed in me when I approached him with my *Anmeldungsbuch* (notice-book). I had committed to memory what I intended to say to him, so that I might not remain dumb on account of my admiration."

During the years following 1807, Gauss developed another peculiarity, *viz.*, reluctance to publish his discoveries. "He was in the habit of letting some of his greatest discoveries remain locked in his drawers for decades, without making them known. For, as he frequently said, he carried on scientific researches for his own sake on account of an innermost call of his soul; and it was a secondary matter to let his works be printed for being made known to a wider circle."* Even when writing out his book, *Disquisitiones Arithmeticae,* Gauss had drafted and redrafted parts of it before sending them to the press; and he used to say that his motto † was

Pauca sed matura,

i. e., "little but mature." But this mental attitude became more and more exaggerated as years went on: with him it became an obsession to give out his discoveries only in the most complete and perfect form possible. The remonstrances of friends were of no avail. For example, the following remarks* addressed by Bessel in 1837 left Gauss unyielding: "You have never recognized your duty to advance the *present* state of knowledge by timely communication of a part of your researches; you live for posterity. The mathematical sciences would have been at present in the whole of Europe and not only in your house, if you had given out what you could give out."

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* See Sartorius von Walterhausen's *Gauss zum Gedächtniss*, p. 78.

† On Gauss's seal was the picture of a tree with few fruits and the words: Pauca sed matura.

By not publishing his discoveries in time, Gauss gave rise to misunderstandings which were otherwise unjustified. For example, Legendre wrote to Jacobi as follows† on receiving from him a communication, dated the 5th August, 1827, to the effect that, according to Schumacher (1780–1850), Gauss was as early as 1808 in possession of the theorems published by Jacobi in the *Astronomische Nachrichten*: "This excess of impudence is not credible in the case of a man who has enough personal merit for not caring to appropriate the discoveries of others." On the 14th April, 1828, Legendre wrote to Jacobi: "There are men like Gauss who make no scruple about ravishing, if they can, the fruit of your researches and about pretending that they have been long in possession of them. Pretension certainly absurd."*

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* See Schering's *Werke, l. c.*, pp. 185–186.

† See Jacobi's *Werke*, Bd. 1, p. 398. The remarks are also quoted in Gauss's *Werke*, Bd. X_2, Abhandlung IV, p. 17.

According to Stäckel and Klein,† the following may *roughly* be taken as a chronological division of the activities of Gauss:—

(1) From his student days at Collegium Carolinum to 1794, numerical calculations which formed the basis of his future investigations on the theory of numbers.

(2) From 1795 to 1800, investigations on the theory of numbers, Algebra and Analysis.

(3) From 1800 to 1820: Astronomy.

(4) From 1820 to 1830: Geodesy.

(5) From 1830 to 1840: Mathematical Physics.

(6) From 1841 onwards: Higher Geodesy, Foundations of Geometry, Geometry Situs, and geometrical representation of complex variables.

While engaged in the study of Geodesy and in actual field work, Gauss also studied the problem of conformal representation and the general theory of surfaces. The discovery of the telegraph was made by him in 1833.‡

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* Jacobi's *Werke*, Bd, 1, p. 418.

† See Klein's *Entwicklung*, *l. c.*, p. 7 and p. 25. Also Stäckel's memoir in Gauss's *Werke*, Bd. X_2, pp. 2–5.

‡ See Klein's *Entwicklung*, *l. c.* p. 20; also Schering's *Werke*, *l. c.*, pp. 188–189.

I proceed now to consider in some detail each of the important writings of Gauss. I shall begin with Gauss's first great work, *Disquisitiones Arithmeticae.*

(I)

The work bears the following dedication* to the Duke of Brunswick which is reproduced here word for word (of course in English garb) on account of its human interest.

"To
His most serene Highness,
Prince Charles William Ferdinand,
Duke of Brunswick and Luneburg.

Your most serene Highness,

I deem it a great happiness to be allowed to place your illustrious name at the head of this work, which I am bound by the sacred duty of filial affection to offer you. For, indeed, most serene Prince, had it not been for your bounty which first gave me access to science, were it not for your continual benefits which unceasingly sustained my studies, I would not have been able to devote myself entirely to mathematics, to which I have always felt myself drawn with an irresistible ardour. Your kindness alone released me from all other cares, and enabled me to devote my time especially to this present work; and thus I have been able not only to undertake my researches—a part of which is contained in this book—but also to pursue them for many years and to commit them to writing. And when I wanted to publish my work, your liberality removed all the obstacles that delayed its publication. It is easier for me to keep in mind, and to admire in silence, the very great and generous interest you have taken in my work, than to praise it in a right and worthy manner. Indeed, not only do I feel myself inferior to such a task, but I also think that no one ignores how great and conspicuous your liberality is towards all those who contribute to the cultivation of science; and that your patronage is equally granted to those branches of science which are commonly believed to be more abstruse and of a less practical utility in ordinary life. For, the depth of your wisdom, which renders you most expert in all that concerns the increase of human prosperity, has thoroughly grasped the necessary connexion existing between all the sciences. And if you, most serene Prince, will consider this book—which is the expression of my gratitude and a token of labour dedicated to the noblest of sciences—to be not unworthy of the signal favour you have for so long bestowed upon me, I shall congratulate myself that my efforts have not been misplaced and that I have come by an honour which I had above all things at heart.

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* See Gauss's *Werke*, Bd. 1, pp. 3–4.

Most serene Prince,
Your Highness's
Most faithfull servant,
C. F. Gauss"

Brunswick, July, 1801.

The book contains, in addition to the dedication a preface of 4 pages, seven sections divided into 366 articles, an addendum of 2 pages and three tables; the total number of pages in the book being 474 as it appears in Gauss's *Werke*, Bd. I. The *first* section (*pp.* 9—13; Arts. 1—12) deals with the congruence of numbers in general. In the *second* section (*pp.* 14—37; Arts. 13—44) are considered congruences of the first degree with one or more unknowns. In the *third* section (*pp.* 38—72; Arts. 45—93) are treated the power residues, *i. e.*, the residues arising from the successive powers of a number. Among other things are given (1) Fermat's theorem that

$$a^{p-1} \equiv 1 \pmod{p},$$

if p be a prime number and a is any number not divisible by p; and (2) Wilson's theorem that

$$(p-1)! \equiv -1 \pmod{p},$$

if p is a prime number. In the *fourth* section (*pp.* 73—119; Arts. 94—152) which treats of congruences of the second degree, quadratic residues and non-residues are first considered, and then two proofs of the reciprocity theorem,* viz.,

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* Gauss gives the theorem naturally not in this form in which Legendre's notation is used; $\left(\frac{q}{p}\right) = +1$ denoting the fact that q can occur as the residue of a square when the modulus is p, $\left(\frac{q}{p}\right) = -1$ denoting that this is not the case (q is a "non-residue" mod. p). The theorem is that $\left(\frac{q}{p}\right)$ and $\left(\frac{p}{q}\right)$ possess the same signs when both are not of the form $4n+3$. The theorem is enunciated by Gauss as follows: If p is a prime

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}},$$

are given.

In this section the quadratic residues and non-residues are thus defined. If

$$x^2 \equiv b \ (\text{mod. } c)$$

be a congruence of the 2nd degree in which c is a prime number and b is an arbitrary positive or negative integer, then the number b is said to be the quadratic residue of the number c if it is possible to find a square x^2 which for mod. c is congruent with b; on the contrary if such a square cannot be found then b is non-residue of c.

If now c be an odd prime number, then must

$$b^{\frac{c-1}{2}} \equiv 1 \ (\text{mod. } c)$$

in order that the congruence $x^2 \equiv b$ (mod. c) should hold.

Since, however, $b^{c-1} = \left(b^{\frac{c-1}{2}}\right)^2 \equiv 1$ (mod. c),

therefore is $b^{\frac{c-1}{2}} \equiv +1$ or -1 (mod. c);
consequently is b quadratic residue or non-residue of c according as $b^{\frac{c-1}{2}} \equiv +1$ or $b^{\frac{c-1}{2}} \equiv -1$ (mod. c).

The *fifth* section (*pp.* 379, Arts. 153—307) is the

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number of the form $4n+1$, then is $+p$, if however p is a prime number of the form $4n+3$, then is $-p$, quadratic residue or non-residue of every prime number which taken positive is quadratic residue or non-residue of p.

biggest. It deals with the forms and indeterminate equations of the second degree. The *sixth* section (*pp.* 380—411 ; Arts. 308—334) and the *seventh* section (*pp.* 412—466 ; Arts. 335—366) deal respectively with various applications of the first five sections and with cyclotomy. "With the appearance of the *Disquisitiones Arithmeticae* begins a new chapter for the Theory of Numbers and the book marks an epoch in the history of Mathematics. But the abstract and synthetic exposition produced the result that the work remained in the beginning a book with seven seals. Gauss himself said once that in the first 20 years it will not be understood. Only the last discovery contained in it on the subject of cyclotomy drew attention to itself. After several decades the book was the mine out of which almost inexhaustibly the greatest mathematicians of our time won new shining material for the perfection of our science."* Klein† has expressed himself about the book in language similar to that in the above quotation. According to him, it was first by the lectures of Dirichlet on the Theory of Numbers that Gauss's method of thought and exposition came to be understood.

(II)

The next publication of Gauss in importance and sequence of time is *Theoria Motus*. In Gauss's *Werke*

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* See Gerhardt's *Geschichte der Mathematik in Deutschland*, 1877, pp. 222—223.

† *l. c*, p 27.

Bd. VII, it occupies 280 pages of which the title page and the preface take 10 pages, the text 250 pages and three tables 20 pages. The work is divided into two books each containing four sections. They are headed as follows:—

LIBER PRIMUS

(First Book)

Relationes generales inter quantitates, per quas corporum coelestium motus circa Solem definiuntur.

(General relations between the quantities which determine the movements of the celestial bodies round the sun.)

Sectio I. Relationes ad locum simplicem in orbita spectantes.

(Part I. Relations referring to one single position on the orbit.)

Sectio II. Relationes ad locum simplicem in spatio spectantes.

(Part II. Relations referring to one single position in space.)

Sectio III. Relationes inter locos plures in orbita.

(Part III. Relations between different positions on the orbit.)

Sectio IV. Relationes inter locos plures in spatio.

(Part IV. Relations between different positions in space.)

LIBER SECUNDUS

(Second Book)

Sectio I. Determinatio orbitae e tribus observationibus completis.

(Part I. Determination of the orbit from three complete observations.)

Sectio II. Determinatio orbitae e quatuor observationibus, quarum duae tantum completae sunt.

(Part II. Determination of the orbit from four observations of which only two are complete.)

Sectio III. Determinatio orbitae observationibus quotanque quam proxime satisfacientis.

(Part III. Determination of the orbit by as many observations as give a close approximation of the orbit.)

Secio IV. De determinatione orbitarum habita ratione perturbationum.

(Part IV. On the determination of orbits, taking into account the perturbations.)

The genesis of *Theoria Motus* is to be found in the investigations of Gauss in relation to the planets Ceres and Pallas. "The first day of the year 1801 was signalized by the discovery of the planet Ceres at Palermo by Piazzi*, and before the first observations of the discoverer—only two in number—had been made known to astronomers, the planet had ceased to be observable from its proximity to the Sun. The planet Uranus had been discovered 20 years before, when near opposition. This was a critical position, which at once gave a near approximation to the elements of his orbit: a stationary elongation of Ceres, though less fertile in its results, was sufficient to assign her such a place between Mars and Jupiter as was required to satisfy Bode's law...

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* (1746 -1826).

... ...

"The complete determination, however, of the elements of a planet's orbit from three geocentric longitudes and latitudes—or from four of the first and two of the second in those cases where the latitudes are evanescent or small—was still a new problem which had been completely solved in the case of comets moving in parabolic orbits, and which Newton, to whom its first solution was due, had pronounced to be *problema omnium longe difficillimum.*

"It was not until the month of October following the discovery of Ceres that Gauss came into the possession of the requisite observations, and in the course of a few weeks he had determined the elements of her orbit with an accuracy fully commensurate with the observations; so much so indeed that the Baron von Zach* was enabled to rediscover the planet at the very first attempt which he made for that purpose on the 7th of

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* (1754–1832).

December, 1801. The elements of Pallas, Juno and Vesta, the discovery of which followed that of Ceres at no great distance of time, were promptly determined by Gauss by methods substantiatly the same, but naturally improved by new artifices and adaptations of formulae which an enlarged study and application had enabled him to give them.

"The *Theoria Motus* contains not only the exposition of these methods and their detailed exemplifications, but a most elaborate discussion of the various problems which present themselves in the determination of the movements of planets and comets from observations made on them under any circumstances."*

Klein's opinion about the book is that it is a great work which, because of the model way in which problems of celestial mechanics are treated there and made amenable to numerical work, has become for calculating or practical Astronomy exactly a law book.

"The methods which Gauss applied have naturally been since his time further developed and made sharper, just as he was not even without predecessors in this field. To give only two names, as a predecessor Lagrange and as a successor Gibbs may be mentioned."†

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* *Proc. Royal Society of London*, 1856, pp. 591–592.

† *l. c.*, p. 9.

(III)

"Disquisitiones generales circa seriem infinitam

$$1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2$$

$$+\frac{\alpha(\alpha+1)(\alpha+2)\cdot\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}x^3+\ldots\text{"}, \text{ Pars Prior,}$$

appears in Gauss's *Werke*, Bd. 3 and covers pp. 125–160 of which the last 2 pages give the values of $\log \Pi(z)$ and $\psi(z)$ for fractional values of z from 0 to 1 in arithmetical progression with common difference .01. The memoir is divided into

Introductio ;

(Introduction)

Sectio Prima (Relationes inter functiones contigues) ;

[Part I (Relations between contiguous functions)]

Sectio Secunda (Fractiones continuae) ;

[Part II (Continued fractions)]

Sectio Tertia (De summa seriei nostrae statuendo elementum quartum $=1$, ubi simul quaedum aliae functiones transscendentes discutiuntur).

[Part III (On the sum of the above-mentioned series putting the required element $=1$; together with the discussion of some other transcendental functions)]

A fairly long analysis of the memoir was given by Gauss himself,* beginning with the following lines : "The logarithmic and circular functions, as the simplest kinds of the transcendental functions, are those with which the analysts have most engaged themselves. They deserved this honour because of their continued penetration into almost all mathematical investigations, theoretical and practical, as well as because of the almost inexhaustible riches in interesting truths which their theory offers. Much less up to now have other transcendental functions been worked at which do not admit of being referred back to those (the logarithmic and circular functions) but must be considered as forming separate higher classes by themselves which only in special cases are connected with the latter. And still many such functions are not less fruitful in interesting relations and consequently not less important than those which Analysis honours because of themselves; also the frequent occurence of the higher functions in many kinds of other investigations must be a recommendation to the man who eagerly asks first for practical uses. Professor Gauss has already been engaged for many years with investigations of those kinds of higher transcendentals, the results of which investigations confirm what has been said above. A comparatively small part of those investigations, which part can be looked upon as an introduction to a series of memoirs to be communicated in future," has been embodied in the present memoir.

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* *Gött. Anzeigen*, 1812 ; also *Werke*, Bd. 3, pp. 197–202.

In order to give a test of the applicability of the series, 23 different algebraical, logarithmic and trigonometrical functions have been given in the *introduction* in terms of the series; and there also it is shown how higher transcendental functions, e. g. the coefficients of cosines of multiples of ϕ in the expansion of $(a^2+b^2-2ab\cos\phi)^n$, can be expressed in terms of the series. "The *first* section deals with the relations between such series of the above form in which the values of one of the first three elements differ by unity, the values of the remaining three remaining equal. Such series are called by the author *series contigue* which may be translated into *allied series*. To each series $F(\alpha, \beta, \gamma, x)$ correspond six allied series, viz., $F(\alpha+1, \beta, \gamma, x)$, $F(\alpha-1, \beta, \gamma, x)$, $F(\alpha, \beta+1, \gamma, x)$, $F(\alpha, \beta-1, \gamma, x)$, $F(\alpha, \beta, \gamma+1, x)$, $F(\alpha, \beta, \gamma-1, x)$, and it is shown here that between the first and any two of the allied series, a linear equation exists. Fifteen equations arise in this manner. Hence follows the important theorem, that if $\alpha'-\alpha$, $\alpha''-\alpha$, $\beta'-\beta$, $\beta''-\beta$, $\gamma'-\gamma$, $\gamma''-\gamma$, are integral numbers, then also between $F(\alpha, \beta, \gamma, x)$, $F(\alpha', \beta', \gamma', x)$, $F(\alpha'', \beta'', \gamma'', x)$ a linear relation holds, and therefore, speaking generally, from the values of two of these functions the value of the third can be deduced. Some of the simplest or otherwise remarkable cases have been particularly collected here by the author."

"The *second* section gives transformations into continued fractions, and indeed for the quotients $\frac{F(\alpha, \beta+1, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)}$, $\frac{F(\alpha, \beta+1, \gamma, x)}{F(\alpha, \beta, \gamma, x)}$, $\frac{F(\alpha-1, \beta+1, x)}{F(\alpha, \beta, \gamma, x)}$, to which further three others by the obviously permissible interchange of the first two elements can be referred. Of these theorems, nearly all the hitherto known developments in continued fractions are only particular cases." Specially remarkable is the case in which $\beta=0$ is put in the expression $\frac{F(\alpha, \beta+1, \gamma, x)}{F(\alpha, \beta, \gamma, x)}$

We have than

$$F(\alpha, 1, \gamma, x)=\cfrac{1}{1-\cfrac{ax}{1-\cfrac{bx}{1-\cfrac{cx}{1-\cfrac{dx}{\cdots\cdots}}}}}$$

where a, b, c, d, etc., are given by the following rules:

$$a=\frac{\alpha}{\gamma},\ b=\frac{\gamma-\alpha}{\gamma(\gamma+1)},\ c=\frac{(\alpha+1)\gamma}{(\gamma+1)(\gamma+2)},$$

$$d=\frac{2(-\alpha+\gamma+1)}{(\gamma+2)(\gamma+3)},\ e=\frac{(\gamma+2)(\gamma+1)}{(\gamma+3)(\gamma+4)},\ \text{etc.}$$

According to the above, the power of a binomial, the series for $\log(1+x)$, for $\log\frac{1+x}{1-x}$, for exponential quantities, etc., can be changed into continued fractions.

The greatest part of the memoir is taken by the *third* section in which are treated the values of the series when the fourth element is put equal to 1. "After it has been proved with mathematical rigour that the series for $x=1$ converges to a finite sum only then when $\gamma-\alpha-\beta$ is a positive quantity, the author refers this sum or $F(\alpha, \beta, \gamma, 1)$ to the expression

$$\frac{\Pi(\gamma-1)\,\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\,\Pi(\gamma-\beta-1)},$$

where the characteristic Π denotes a special kind of transcendental function, whose generation is based by the author on an infinite product. This function, most important in Analysis, is nothing other than Euler's function,

$$\Pi(z) = 1.\ 2.\ 3.\ 4.\ \ldots\ z;$$

only this kind of generation or definition is, according to the author's verdict, thoroughly untenable, because it has a clear meaning only when z has positive integral values." The section contains a number of remarkable properties of the function $\Pi(z)$ as also of the function,

$$\frac{d \log \Pi(z)}{dz},$$

which the author denotes by $\psi(z)$. It is proved, for example, that $\psi(z)-\psi(0)$ can be expressed in terms of logarithms and circular functions in case z is a rational number. $\psi(0)$ is the known Euler's constant ·5772156649....

(IV)

"Methodus nova integralum valores per approximationem inveniendi." (*Gött. Abh.*, 1814; also *Werke*, Bd. 3, pp. 165–196).

At the time of Gauss, the most commonly used method for finding the approximate value of an integral $\int_a^b f(x)\,dx$ was that given by Newton and Cotes, viz, to take points $a, a+h, a+2h$, etc., h being $\frac{b-a}{n}$, and, denoting the corresponding co-ordinates by y_1, y_2, y_3, etc., to consider

$$h\,(A_1y_1+A_2y_2+A_3y_3+\ldots)$$

as the approximate value, the A's being certain constants with the property that the coefficients at equal distances from the beginning and end are equal. For $n=4$, Newton took

$$A_1=\tfrac{1}{8},\ A_2=A_3=\tfrac{3}{8},\ A_4=\tfrac{1}{8}.$$

In the paper in question, Gauss studies carefully the theory of the coefficients A's first and the degree of exactitude of the result. "It is found that the conditions on which this degree of exactitude depends are of the kind that one can double the exactitude by choosing the same number of coefficients suitably at unequal distances, so that one, with an arbitrary number of suitably chosen ordinates, goes as far as with the double number of ordinates at equal distances. The chief contents of the paper are the aforesaid investigation together with the complete theory of the most expedient choice of the ordinates, the coefficients to be used therein and the determination of the degree of exactitude which the method gives."*

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* *Gött. Anzeigen*, 1814; also *Werke*, Bd. 3, pp. 202—206.

(V)

"Theoria combinationis observationum erroribus minimis obnoxiae," in three parts. (*Gött. Abh.*, 1823 and 1826).

In his analysis of the memoir, Gauss himself says*: "The author of the present memoir, who in 1797 first investigated this problem according to the fundamental theorems of the Calculus of Probability, soon found that the determination of the most probable value of the unknown quantity is impossible, if the function which represents the probability of the error is not known. In so far as this is not known, nothing remains but to take such a function hypothetically. It appeared to the author most natural, first to strike the inverse way and to seek the function which must be laid at the base, if a general rule, recognized as good, is to flow out of it, namely the rule that the most probable value of an unknown quantity, whose values by means of several observations of equal reliability are known, is the arithmetic mean of those values. It followed that the probability of an error x must be taken to be proportional to an exponential quantity of the form e^{-hx^2}, and that then exactly that method will become generally necessary at which the author had arrived many years before by means of other considerations. This method, which the author had the occasion to apply almost daily since 1801 in all kinds of astronomical calculations is now in general use under the name of the method of least squares: and its foundation by the Calculus of Probability as well as the determination of the exactitude of the result itself, together with the other investigations connected therewith, have been fully developed in the *Theoria Motus Corporum Coelestium*." The first two parts of the memoir contain 40 paragraphs and the supplement includes 25 paragraphs. In this connection may be mentioned a short paper by Gauss entitled "Determination of the exactitude of observations", published 5 years before the writing of the *Theoria combinationis*.

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* *Gött Anzeigen*; also *Werke*, Bd 4.

(VI)

"Allgemeine Auflösung der Aufgabe die Theile einer gegebenen Fläche auf einer anderen gegebenen Fläche so abzubilden dass die Abbildung dem abgebildeten in den kleinsten Theilen ähnlich wird."

This memoir carries with it the following preface, dated December, 1822. "The author of this memoir has believed that he must consider the repeated choice of the problem which forms the subject of the memoir as a proof of the importance which the Royal Society (of Denmark) attaches to it, and is therefore encouraged to submit his solution which he discovered long ago, from which submission he would have been otherwise kept away on account of his receiving late the knowledge of the prize question. He regrets that the latter circumstance has necessitated his confining himself almost only to the essentials and to the indications of certain uses for maps and higher Geodesy, since without the nearness of the last date for submission he would have gladly worked out further some auxiliary circumstances and would have worked out comprehensively the many-sided applications to higher Geodesy which he must now reserve for another time and another place." The memoir covers pp. 193—216 of Gauss's *Werke*, Bd. IV. The immediate predecessor of Gauss in this field of research was Lagrange who had shown in 1781 how one would obtain the conformal representation of any surface of revolution,

$$ds^2 = dp^2 + G(p)\, d\lambda^2,$$

on the x, y—plane. In fact, Lagrange's treatment leads to the result that, for the conformal representation of

$$ds^2 = \psi(\lambda, \mu).\ (d\lambda^2 + d\mu^2)$$

on the x, y—plane, it is necessary and sufficient that

$$x + \iota y \text{ be a function of } \lambda + \iota\mu,$$
$$x - \iota y \text{ be a function of } \lambda - \iota\mu,$$

where λ, μ satisfy the condition

$$dx^2 + dy^2 = \phi(\lambda, \mu)\ (d\lambda^2 + d\mu^2).$$

Gauss, to whom the introduction of the term "conform" is due, showed how one can bring the form,

$$ds^2 = E\, dp^2 + 2F\, dp\, dq + G\, dq^2$$

for the square of the line-element of the general curved surface, to the special form

$$ds^2=\phi(\lambda, \mu)(d\lambda^2+d\mu^2).$$

After giving the analytical form of the problem, Gauss gives a number of examples and concludes his paper with the consideration of the special case of the conformal representation of an ellipsoidal surface on a sphere. It is noteworthy that the name of Lagrange is not mentioned even once in the memoir.

(VII)

"Desquisitiones generales circa superficies curvas" (*Gött. Abh.*, 1828; *Werke*, Bd. 4).

"The theory of curved surfaces was founded in the works of Euler, Monge, Dupin, Cauchy and others, originating from the 18th century and the beginning of the 19th century, almost exclusively on the supposition that the surface under consideration is given by *an* equation between the coordinates x, y, z. With preference the equation is taken in the form $z=f(x,y)$. The parametric representation of a surface, even if not unknown before, came first in use after C. F. Gauss had shown, by his prize-memoir, written in 1822 on the conformal representation of two surfaces on one another, and by his *Disquisitiones generales circa superficies curvas* finished in 1827, what advantages one can derive if one regards the surfaces not as limits of bodies but as bodies whose one dimension vanishes, and at the same time as flexible but not extensible."*

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* See *Gött. Anzeigen*, 1828.

The memoir contains 29 articles. After defining the measure of curvature at a point of the surface with the help of an auxiliary sphere, Gauss gives in the end of Art. 11, the expression for the measure of curvature in terms of E, F, G and their differential coefficients of the first two orders. In Art. 12, he deduces a number of results from the above expression, including the theorem that by deformation the measure of curvature at a point is unchanged. The remaining part of the memoir is devoted mainly to the theory of the triangles formed by shortest lines. Theorems of this theory include the following :—If a, b, c be the sides of such a triangle, A, B, C the opposite angles, σ the area of the triangle, then up to quantities of the 4th order (a, b, c being considered to be of the 1st order), $\frac{1}{3}(\alpha+\beta+\gamma)\sigma$ is the excess of the sum $A+B+C$ over two right angles, α, β, γ being the curvatures at the vertices of the triangle. Also the angles of a plane rectilinear triangle whose sides are a, b, c are

$$A-\tfrac{1}{12}(2\alpha+\beta+\gamma)\sigma,$$
$$B-\tfrac{1}{12}(\alpha+2\beta+\gamma)\sigma,$$
$$C-\tfrac{1}{12}(\alpha+\beta+2\gamma)\sigma.$$

The above three expressions are given in Art. 28.

(VIII)

"Theoria attractionis corporum sphaeroidicorum homogeneorum." (*Gött. Abh.*, 1813 ; *Werke*, Bd. 5).

The triple integrals giving the components of attraction are first transformed into surface integrals

(See theorems III and VI); then appropriate coördinates on the surface are introduced to change the surface integrals suitably for further treatment. Comparing the values for two confocal ellipsoids, we have the theorem of Maclaurin: the potentials of confocal ellipsoids on one and the same external point behave as their masses. Lastly, the surface integrals are reduced to simple integrals and thus the final results are obtained as elliptic integrals. In an analytical review by himself,* Gauss claims that his memoir gives for the first time the *proper* method of solving the problem of finding the components of attraction of an ellipsoid, and that it is the direct method which to Lagrange appeared such a desideratum when he wrote in 1793 about the methods of Legendre and Laplace: "One may regard their solutions as the chefs d' oeuvres of analysis; but one may desire still a more direct and more simple solution; and the continual progress of analysis gives reasons for hoping for such a solution."

(IX)

"Bestimmung des Breitunterschiedes zwischen den Sternwarten von Göttingen und Altona", 1828.

This work is in four parts. The first part gives the list of the stars observed. In the second part are all the 900 observations stated which were made at

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* *Gött. Anzeigen*, 1813.

Göttingen from the beginning of April, 1827, to the middle of May and later at Altona during June. The third part develops the results from the observations. The chief result is the difference in latitude which is fixed at 2° 0′ 57″.42 with a possible error of 0″.07. The fourth part gives the latitude of Seeberg.

Up to the time of these observations, the triangle formed by the three mountain summits, Hoher Hagen, Brocken, Inselberg, was the greatest geodetically worked triangle. Also in the observations the instrument Heliotrop, discovered by Gauss, was made much use of.

(X)

"Untersuchungen über Gegenstände der höheren Geodäsie," in two parts. (*Gött. Abh.*, 1844 and 1847). also *Werke*, Bd. 9.)

Klein* says of this work of Gauss: "He has given the methods and models in accordance with which measuring or practical Geodesy even to-day exclusively works. As a most important point is to be mentioned above all the consistent application of the method of least squares. A speciality of Gauss is the use of a definite conformal projection of the ellipsoid on the plane, which Gauss himself calculated...... up to the slightest deviation of geodesic lines from straight lines. These two applications of thought and the way they have been carried out, have had such definite influence on the further development of the science of Surveying that surveyors count Gauss as one of them"

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* Klein's *Entwicklung*, l. c. p. 15.

(XI)

"Determinatio attractionis quam in punctum quodvis positionis datae exerceret planeta, si eius massa per totam orbitam ratione temporis, quo singulae partes describuntur, uniformiter esset dispertita. (*Göt. Abh.*, 1818; *Werke*, Bd. 3, pp. 331–357)

"Determination of the attraction which a planet would exert on any point, if the planetary mass were uniformly distributed over its whole orbit according to the time in which each portion of the orbit is described." (*Gött. Abh.*, 1818)]

The character of the memoir may be recognized by the title. This is how Gauss pictures to himself the secular disturbance due to a planet: "Let the mass of the planet be distributed along its orbit and indeed in inverse proportion to the velocity at the point; then the attraction of this ring on another body is exactly the magnitude of the secular disturbance. This conception is an example of the peculiarly plastic mode of thinking which always comes to the fore in the case of Gauss. He was not only the calculator, who triumphantly overcame all difficulties, but his numbers were for him full of life and readily admitted of being bound to perceptive representations"*

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* Klein's *Entwicklung*, p. 13.

(XII)

"Principia generalia theoriae figurae fluiodorum in statu equilibrii." (*Gött. Abh.*, 1829; also *Werke*, Bd. 5, pp. 29–77)

Gauss* says: "The subject of the paper belongs to that domain of mathematical Physics which Laplace opened in a brilliant manner by his two epoch-making papers, 'Theoria de l'action capillaire' and 'Supplement à la theorie de l' action capillaire.' In this work are laid down completely for the first time in the history of the Calculus of Variations, for a two-dimensional variation-problem with variable integration-region, the conditions following from the vanishing of the first variation; thus is given not only the partial differential equation of the problem but in particular also the corresponding boundary condition on the limits. From the principle of virtual displacements, the problem of the equilibrium is transformed into an isoperimetrical problem in two dimensions with variable integration-region. As, soon after the publication of this memoir, a series of comprehensive and important works on the general theory of the extrema of double and multiple integrals came out, namely, those of Bordoni (1831), Poisson (Nov., 1831), Pagani (Dec. 1834); Ostrogradsky (Jan., 1834), Delaunay (sent to the French Institute before April, 1842), Cauchy (1844), Sarrus (sent before April, 1842),

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* *Gött. Anzeigen*, 1829. Also *Werke*, Bd. 5, p. 287.

Lindelöf (1856), it is not wrong to presume that Gauss's memoir gave the impulse to this whole development.

(XIII and XIV)

"Intensitas vis magneticae terrestris ad mensurum absolutam revocata." (*Gött. Abh.*, 1829; also *Werke*, Bd. 5, pp. 81-118).

"Allgemeine Theorie des Erdmagnetismus", 1838—39. (*Werke*, Bd. 5, pp. 121—193).

In the first memoir which is of 28 articles, Gauss first points out how, of the three elements, declination, inclination and intensity, defining the state of terrestrial magnetism at a place, intensity, which had only very recently begun to be observed, could not be rigorously given by the current method of taking it to vary as the square of the period of vibration of one and the same magnetic needle. He points out that the period depends on three quantities, viz. the intensity, the static moment of free magnetism in the needle and the moment of inertia of the needle. The memoir gives (1) the mathematical theory, (2) different experimental methods peculiar to the author, e. g., for finding the intensity independently of the moment of inertia of the vibrating needle, and (3) various attempts at determining the local intensity.

The second memoir consists of 40 articles, a supplement, two figures and severel tables. It is one of the most important achievements of Gauss who develops in it the general theory of terrestrial magnetism independently of all special hypotheses on the distribution of magnetic fluid in the body of the Earth and then gives the results obtained by the first application of the methods.

The title of the second memoir may be misleading, it does not deal with a physical theory but only with a representation of observed results by means of Spherical Harmonics and interpolation.* "Gauss represents the potential of the earth-magnetism by means of a finite series of Spherical Harmonics and indeed he goes up to the 4th order:

$$V=\frac{P_1}{r^2}+\frac{P_2}{r^3}+\frac{P_3}{r^4}+\frac{P_4}{r^5},$$

where P_n is a homogeneous polynomial which satisfies $\nabla^2 P_n=0$. Such a polynomial has $2n+1$ independent constants, so that for the above representation the calculation of 24 constants is necessary. From this series the forces X, Y, Z are derived by differentiation." These constants are fully given in his *Werke*, Bd. 5, pp. 150, 151.

(XV)

"Allgemeine Lehrsätze in Beziehung auf die im vekehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs-und Abstossungs-Kräfte" 1839—1840. (*Werke*, Bd. 5, pp. 197—242)

This memoir of 37 articles may be looked upon as an introductory treatise on the theory of potential and is remarkable not only for its originality but also for the rigour of its treatment. Some of its most important contents may be enumerated as follow:—In Arts. 1—5 are given the names, *potential*, *equipotential surface*, and the proof of what Gauss calls the known equation,

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* See Klein's *Entwicklung*, l. c., pp. 21-22.

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

without mentioning Laplace's name. In Arts. 6—11 is considered V at a point inside the attracting or repelling mass and the equation of Poisson is established rigorously on the sole assumption of the density possessing the partial differential coefficients of the first order. In Arts. 12—18 the surface potential is considered and its properties are given. At the end of Art. 18, Gauss says: "The theorems expounded up to now are, as regards their essential contents, not new, but could not have been passed over because of their connection as the necessary preliminaries to the following investigations in which a series of new theorems will be developed." In Arts. 19—28 are found amongst others, the following theorems :—

$$(1) \quad \int V dS = 4\pi\,(RM_0 + R^2V_c),$$

the integration being extended over a sphere of radius R, at the centre of which V_0 is the potential of the mass outside the sphere, and M_0 being the mass inside the sphere.

(2) The potential outside an attracting mass is neither a maximum nor a minimum.

(3) The potential V of masses, which are all outside a connected space, cannot in a part of this space have a constant value and at the same time in another part of that space have a different value.

(4) $$\int \frac{\partial V}{\partial n} d\,S = 4\pi\,M,$$

M being the total mass inside the surface.

(5) $$\int V \frac{\partial V}{\partial n} dS = -\int q^2 dT,$$

q being the force and d T an element of the volume bounded by the surface S; there being no mass inside S.

(6) If, of masses which are purely inside the finite space T or are wholly or partly distributed on the surface S in a continuous manner, the potential in all the points of S has a constant value, then the potential in every point of the external infinite space will be firstly 0, if $A=0$ and secondly less than A, if $A \neq 0$.

In Arts 29—37, the chief result is the following existence theorem which is essentially Dirichlet's principle. There always exists a non-uniform distribution of mass, if a uniform distribution is not possible, for which

$$W=V-U$$

has a constant value in all the points of the surface, U being a continuous function of the co-ordinates of the surface,

(XVI)

Doctor-dissertation of 1799 and three more proofs of the fundamental theorem of Algebra given in 1815, 1816 and 1849.

(XVII)

"Theorie des arithmetisch-geometrischen Mittels und der Modulfunktion."

Schlesinger has given, after Gauss's treatment, an outline* of the theory of arithmetico-geometric means. The following account of Gauss's achievement in this field is based on Schlesinger's outline:—

Definition: Let $a \neq 0$, $b \neq 0$, $a^2 \neq b^2$. If we form the algorithm (already used by Lagrange, see *Oeuvres*, t. 2, p. 251, p. 272, p. 304)

$$a_1 = \frac{a+b}{2}, \quad b_1 = \sqrt{(ab)},$$

$$a_2 = \frac{a_1+b_1}{2}, \quad b_2 = \sqrt{(a_1 b_1)},$$

. . . .

$$a_{n+1} = \frac{a_n+b_n}{2}, \quad b_{n+1} = \sqrt{(a_n b_n)},$$

there exists $\lim\limits_{n=\infty} a_n$ as well as $\lim\limits_{n=\infty} b_n$; and each is called M$(a, b)$ or ag M, the arithmetico-geometric mean between a and b.

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* "Abriss der Theorie des arithmetisch-geometrischen Mittels (Gauss's *Werke*, Bd. X, Erste Abitheilung, pp. 251–257).

(A) *Properties.*

(*a*) For an arbitrary ρ,

$$M(\rho a, \rho b) = \rho M(a, b).$$

(β) Let $k = \frac{c}{a}$, where $c = \sqrt{(a^2 - b^2)}$;

$$k' = \frac{b}{a}, \text{ so that } k^2 + k'^2 = 1;$$

then

$$\frac{1}{M(1, k)} = F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k'^2), \; |k'| < 1,$$

and

$$\frac{1}{M(1, k')} = F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k^2), \; |k| < 1.$$

(γ) $$\frac{1}{c^2}\, d\left(\log \frac{a}{b}\right)$$

$$= -\frac{\pi}{2} \frac{1}{\{M(a, b)\}^2}\, d\left\{\frac{M(a, b)}{M(a, c)}\right\};$$

this was called by Gauss the beautiful theorem and regarded as very important.

(δ) $$\frac{1}{M(1, k)} = \frac{2}{\pi} \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}},$$

$$\frac{1}{M(1, k')} = \frac{2}{\pi} \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

(B) *Klein's opinion* about Gauss as the discoverer of the theory of elliptic functions.*

On March 19, 1797, as the item 60 in Gauss's diary shows, Gauss had recognized the double periodicity of the lemniscate integral

$$\int \frac{x}{\sqrt{(1-x^4)}} .$$

"On May 30, 1799, (see No. 98 of the diary) he arrived at a highly significant result: he finds the connection of the arithmetico-geometric mean with the length of the lemniscate, and indeed by a way purely of calculation, whilst he determines the value of $\frac{1}{M(1, \sqrt{2})}$ up to 11 decimal places. Without still recognizing clearly the situation, he sees the bearing of this discovery which 'opens an entirely new field of Analysis'. From now onwards the development goes rapidly forwards in the region of elliptic functions. At first he engages himself with the lemniscate function, i. e., with the special case of a square period-parallelogram. The numbers 105-109 (of the diary) from May 6 to June 3, 1800, however, detail the discovery of the general doubly-periodic functions. The square is replaced by the general parallelogram. Thereby is the complete theory of the general elliptic functions and of the modulfunctions created and by one stroke the development is made quite in advance of Abel and Jacobi."

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* See his *Entwicklung*, l. c., pp. 33-34.

(C) *Some quotations from Gauss's own statements relating to the works of Abel and Jacobi.* (See also *Werke*, X_1, p. 278.)

To Schumacher in April, 1816*: "Have you then really forgotten that the arithmetico-geometric mean, with which Herr Degen occupies himself, is quite the same as that with which I have occupied myself since 1791 and could write on that a quarto volume of moderate size?"

To Bessel on March 30, 1828†: "To write out the investigations, set on foot in 1798, I shall not be able, because many things have to be cleared up first. Herr Abel has, as I see, now anticipated me and spares me the trouble in relation to about ⅓ of these matters, specially since he has made all developments with conciseness and elegance. He has taken the same way as I struck in 1798; therefore the great coincidence of the results is not to be wondered at. To my wonder, this stretches itself indeed to the form and partly to the choice of symbols, so that many of his formulae appear as pure copies of mine. To prevent any misconception, I note, however, that I do not remember having communicated any thing of these matters to any one".

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* *Werke*, Bd. X_I, p. 247.

† *Ibid*, p. 248.

(XVII)

"Dioptrische Untersuchungen" (*Gött. Abh.* 1840; *Werke*, Bd. 5, pp. 245—276).

This memoir of 22 articles is intended "to show that the theorems (due to Cotes, Euler and Lagrange) can be generalized by taking into account the thickness of the lenses and can still retain their simplicity." For this purpose, the author deems it necessary to expound the fundamentals of Dioptrik in a new manner. Among other things, it is shown how the case of a system of lenses with finite thicknesses can be reduced to the case of a system of infinitely thin lenses.

(XIX)

Foundations of Geometry.

This is what Klein says*: "From 1792 onwards Gauss occupied himself like all his contemporaries with unsuccessful attempts to *prove* the parallel-theorem from the given axioms. He refuted all the "proofs" sent to him by others, for example in the letters to the elder Bolyai of 1799 and 1804 (Bd. 8, pp. 159, 160). During the exposure of all these false conclusions, he turned his thoughts more and more to the positive creation of a non-euclidean Geometry. He can find no contradiction therein. In a letter to Schumacher of 1808 (Bd. 8, p. 165) he indicates that from such a non-euclidean supposition the existence of an absolute unit length in space will follow." In the article "Prinzipien der Geometrie" in the *Encyklopädie der math. Wiss.*, Enriques says[*]: "Gauss appears to have been the first who conceived the impossibility of proving the parallel-postulate and therefore the possibility of a general geometry which disregards it, and he has himself laid down the foundations of such a geometry." Gauss had anticipated both Johann Bolyai[†] (1802 – 1860) and Lobatschefsky (1793 – 1856) who was undoubtedly the first to *publish*[‡] a work on non-euclidean Geometry.

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* *l. c.*, p. 58

XX.

Geometria Situs.

"Of the writings which Gauss published not one refers directly to Geometria Situs, and still this subject engaged his attention all his life. From talks with Gauss in his last years from 1847 to 1855, Sartorius von Walterhausen reports that he had an extraordinary hope of the building up of Geometria Situs in which wide and entirely untraversed fields existed which had not been touched by our present calculus.[§]" An entirely similar statement had been made by Gauss about 50 years before. Gauss's writings, left after his death and published in his *Werke*, Bd. VIII, include on pp. 271—285 remarks on the formation of knots on closed curves. There are also remarks on Geometria Situs to be found in the dissertation, (*Werke*, Vols. 3 and 6). He influenced the researches on the subject by Möbius (1790—1868), Listing (1806—1882) and Riemann.

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[*] p. 41

[†] His work first appeared in 1832 in the Appendix to his father's works.

[‡] in 1829 in a Kasan newspaper.

[§] *Werke* X_2, IV, pp. 46 – 56.

With the publication of *Disquisitiones Arithmeticae* in 1801, Gauss came to be recognized by the mathematical world as the equal of Euler, Lagrange and Legendre, the greatest mathematicians of the 18th century. The publication of his work, *Theoria Motus Corporum Coelestium*, in 1809, raised his reputation still higher and it is related that Laplace considered him even at that time to be the greatest living mathematician in Europe.* The subsequent discoveries of Gauss struck the world with wonder and for a long time he occupied a position of undisputed preeminence. Jacobi compared him to Archimedes† and so careful a critic as Stephen Smith‡ formed the following estimate of him:—"If we except the great name of Newton

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* See Schering's *Werke, l. c.*, p. 235, where it is stated that on **A.** von Humboldt's asking Laplace who the greatest mathematician in Germany was, Laplace mentioned J. F. Pfaff. When v. Humboldt expressed his surprise at Gauss's name not being mentioned, Laplace said that Gauss was the greatest mathematician in the whole world but Pfaff was the greatest only in Germany.

† See Schering's *Werke, l c.*, p. 203.

‡ *Collected Papers*, Vol. 1., pp. LXXIV-LXXV.

(and the exception is one which Gauss himself would have been delighted to make), it is probable that no mathematician of any age or country has ever surpassed Gauss in the combination of an abundant fertility of invention with an absolute rigorousness in demonstration, which the ancient Greeks themselves might have envied. It may be admitted, without any disparagement to the eminence of such great mathematicians as Euler and Cauchy, that they were so overwhelmed with the exhuberant wealth of their own creations, and so fascinated by the interest attaching to the results at which they arrived, that they did not greatly care to expend their time in arranging their ideas in a strictly logical order, or even in establishing by irrefragable proof propositions, which they instinctively felt, and could almost see, to be true. With Gauss the case was otherwise. It may seem paradoxical, but it is probably nevertheless true, that it is precisely the effort after a logical perfection of form which has rendered the writings of Gauss open to the charge of obscurity and unnecessary difficulty The fact is that there is neither obscurity nor difficulty in his writings, as long as we read them in the submissive spirit in which an intelligent school-boy is made to read his Euclid. Every assertion that is made is fully proved, and the assertions succeed one another in a perfectly just and logical order; there is nothing so far of which we can complain. But when we have finished the perusal, we soon begin to feel that our work is but begun, that we are still standing on the threshold of the temple, and that there is a secret which lies behind the veil and is as yet concealed from us. No vestige appears of the process by which the result itself was obtained, perhaps not even a trace of the considerations which suggested the successive steps of the demonstration. Gauss says more than once that, for brevity, he only gives the synthesis, and suppresses the analysis of his propositions. "Pauca sed matura" were the words with which he delighted to describe the character which he endeavoured to impress upon his mathematical writings. If, on the other hand, we turn to a memoir of Euler's, there is a sort of free and luxuriant gracefulness about the whole performance, which tells of the quiet pleasure which Euler must have taken in each step of his work; but we are conscious nevertheless that we are at an immense distance from the severe grandeur of design which is characteristic of all Gauss's greater efforts. The preceding criticism, if just, ought not to appear wholly trivial; for though it is quite true that in any mathematical work the substance is immeasurably more important than the form, yet it cannot be doubted that many mathematical memoirs of our own time suffer greatly (if we may dare to say so) from a certain slovenliness in the mode of presentation; and that (whatever may be the value of their contents) they are stamped with a character of slightness and perishableness, which contrasts strongly with the adamantine solidity and clear hard modelling, which, (we may be sure) will keep the writings of Gauss from being forgotten long after the chief results and methods contained in them have been incorporated in treatises more easily read, and have come to form the common patrimony of all working mathematicians.

... ...

"It is not the greatest, but it is perhaps not the least, of Gauss's claims to the admiration of mathematicians, that, while, fully penetrated with a sense of the vastness of science, he exacted the utmost rigorousness in every part of it, never passed over a difficulty as if it did not exist, and never accepted a theorem as true beyond the limits within which it could actually be demonstrated".

Honours were showered on Gauss by his countrymen and foreigners. In the year 1828, he was described as follows on the title page of his publication mentioned on p. 40 under (IX):

"Knight of the Guelphian and Danneborg orders, Royal British and Hanoverian Court Councillor, Member of the Academies (or Societies) of Berlin, Copenhagen, Edinburgh, Göttingen, London, Munich, Naples, Paris, St. Petersburgh and Stockholm, and of the Italian, American, London and Amsterdam Astronomical Societies". In the year 1855, the King of Hanover had a medal struck in honour of Gauss who was described on the medal as the "Mathematicorum princeps"*.

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* See Klein's *Entwicklung*, l. c., p. 60.

There is unfortunately an aspect of Gauss's character which cannot be passed over, and this is reflected in the attitude adopted by him towards some of his great contemporaries, Cauchy, Abel and Jacobi. This attitude has been variously commented upon. But the following facts remain true: (1) that, although Cauchy was admittedly the first to publish a theory of functions of a complex variable, his name is not mentioned even once in the writings of Gauss; (2) that Gauss, not only offered no encouragement to Abel in his struggle for recognition, but gave expression to an opinion on Abel's first great paper which chilled Abel's feelings towards Gauss for ever; and (3) that the recognition he accorded to Jacobi's work on elliptic functions was rather reluctant. There is no doubt that during the latter half of his life, say after 1816, when he had established himself firmly at Göttingen, he became more and more reserved. His attitude towards Cauchy, who was also ignored by Gauss's protege, Riemann, is difficult to account for; it might have been due to the consciousness of having been the first to build up a theory which Cauchy was the first to publish. Abel's reverent approach with his paper on the impossibility of the solution of the general quintic by radicals alone was spurned by Gauss with the remark that the paper was a monstrousity. Later on, after Abel's sad death, Gauss did express a high opinion of Abel; but in his life-time Abel looked upon Gauss as one not a friend.

Gauss was married twice. By his first wife, Johanna, born Osthoff, to whom he was engaged in November, 1804 and whom he married on the 9th October, 1805, he had three children, viz. (1) his eldest son, Joseph, who was born at Brunswick on the 21st August, 1806, was later Ober—Baurat in Hanover and died in 1873 ; (2) Minna, who was born at Göttingen on the 29th February, 1808, was married to Professor Ewald and died at Tübingen on the 12th August, 1840 ; (3) Louis (Sept., 1809—March, 1810), because of whose birth Gauss lost his wife in October, 1809. Gauss was in great mourning on account of his wife's death and a sheet of paper in which he laments* her is full of his affection and regard for her and of his pain on account of the separation. Perhaps the year 1805 was the happiest year in his long life. In April of the year 1810, Gauss was engaged to Minna Waldeck, daughter of the Göttingen Hofrath, Waldeck, and was married to her in the August of the same year. From this marriage also there were three children, viz. Eugen (1811—1896), Wilhelm (1813—1879) and Therese (1816—1864), of whom the two sons migrated to America and lived there with their numerous descendants. The second wife of Gauss died in 1831. Gauss is believed to have felt greater attachment to his daughters than to his sons with the possible exception of Joseph.

In spite of the great success achieved by Gauss in the fields of Mathematics and the allied sciences, his domestic life was not without some shadows. At times he suffered from hypochondria and went to the length of thinking his life a burden.* For example, during the French occupation of Göttingen and the adjoining territory between 1807 and 1810, he was in pecuniary difficulties because a considerable war contribution was imposed on him which left with him little of the salary of his newly occupied post at Göttingen. He lived in those days in a miserable house. According to Klein*, Gauss's neighbours, and "before all, his family, showed not the least understanding for his gigantic labours which were to them without purpose or aim and which took away his interest from all other things and which did not bring to any one any apparent result. People flung on him the bitterest reproaches and there were persons who doubted his sanity." It was about this time that Gauss wrote on a sheet containing calculations relating to elliptic functions: "Der Tod ist mir lieber als ein solches Leben." † That Gauss was over-sensitive to unpleasant impressions, is evidenced by the following lines written by him to his friend Bolyai in 1848‡: "It is true that my life has been decorated with much that the world holds to be enviable............I will willingly admit that the same fate, which has become so difficult for me and is still so, would be to many others much easier. But the frame of mind belongs to our Ego which the Creator of our existence has given to us and we are able to alter that frame very little." The serenity of mind which was necessary for scientific work, was in his case only too much, and in too many ways, disturbed.

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* See Mack's book, p. 16.

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* See Klein's *Entwicklung*, *l. c.*, p 11–12.

† See Klein *Entwicklung*, *l c.*, p. 11. The words in English translation come to: "Death is dearer to me than such a life."

‡ See *Briefwechsel zwischen Gauss und Bolyai*, p. 132; also Stäckel's memoir "Gauss als Geometer", p. 14

On the morning of the 15th February of the year 1851, Goldschmidt, an old friend and pupil of Gauss was found dead in his bed after he had spent a part of the night in making observations at the Observatory and showing some foreigners the Pleiades through a telescope. This sudden death produced a deep impression on Gauss who had always held Goldschmidt in the highest respect. "In the two following winters Gauss complained repeatedly about his health; he suffered partly from sleeplessness and shortness of breath and partly from dyspepsia which he considered to be his chief trouble. In consequence of this indisposition he had become used to rise every night at about 3 o'clock and to take seltzer water and warm milk; it was a simple remedy which seemed to relieve his heaviness of stomach. Up to that time he had used two medicines prescribed by Olbers about 40 years before; otherwise the use of medicine was completely unknown to him. On account of the extremely little trust which he had in medical science for any possible help, he could not for a long time decide to ask for the advice of a physician during his illness."* Finally, on repeated requests by his most intimate friends and relatives he agreed to consult his friend and colleague, Baum, who came to him as a physician for the first time on the 21st January, 1854. After an investigation extending over several days, Baum diagnosed the disease of Gauss as a dilatation of the heart and expressed little hope of recovery or of the patient's life being prolonged for any considerable time. Olbers had seen signs of this disease in Gauss much earlier and had prescribed certain rules for his observance as a precaution. The application of suitable remedies as well as the improvement in the season appeared to have a favourable effect on Gauss so that in the summer he was able to attend regularly between 11 A. M. and 1 P. M. at the literary museum and to take short walks.

On the 16th June, 1854, he went out of Göttingen, for the first time in more than 20 years, to visit the railway that was being constructed between Cassel and Göttingen. Unfortunately on account of a passing locomotive, the horses of his carriage became restive and overturned it with the result that the driver was severely injured. Gauss himself was not at all hurt and returned with his daughter to the Observatory. This narrow escape, followed by the death of his old and honoured friend, Lindenau, produced a deep impression on him. He spoke frequently to his friends about Lindenau's noble qualities but apparently hoped to live for some years more. On the 31st July, 1854, the opening of the railway was celebrated at Göttingen and Gauss came to the city to see the festivities: this was his last day of fairly good health.

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* Sartorius von Walterhausen: *Gauss zum Gedächtniss*, p. 71. The whole of the following account of Gauss's last days is taken from the same book.

With the approaching autumn the illness increased and there appeared a swelling of the feet which had been now and then slightly perceptible long before. On account of shortness of breath he was unable to walk to the literary museum, and even in his own house he could take only a few steps and that with great difficulty. Although in the last year of his life and unable to work continuously, Gauss even then read much and wrote nearly every day but with greater trouble. In the first week of January, 1855, although Gauss felt himself to be still suffering, he hoped to recover his health as a letter to Praël, which he wrote on the 5th January, showed. Von Walterhausen says that, for the first time, his hand-writing, which had been up to then so beautiful, showed signs of shaking. After the illness had gone up and down several times, von Walterhausen saw Gauss again on the 14th January and found also with him the sculptor, Hesemann, who had come in accordance with the orders of the King of Hanover to prepare a medaillon of the great mathematician. The last visit of v. Walterhausen was on the 21st February soon after noon when he found Gauss strikingly altered but still fully conscious. On the 22nd afternoon he had his last severe fight with death and about the evening be seemed to be better. He had not lost consciousness, heard all that was being said and enquired about those present in the room. He died in great tranquility on the morning of the 23rd February, 1855. On the 26th February, his body, which had lain in state in the Rotunda of the Observatory, was carried to the grave followed by a vast multitude of admirers, friends and students.

On his death he left one surviving son* by his first wife, and by his second wife he left two sons† and a daughter, Therese, to whom he was much attached and who looked after him in his last years with the greatest affection and care. Apart from his individuality as a scholar and researcher, the character of Gauss as a man deserves special notice. Sartorius von Walterhausen, who was a devoted pupil and friend of Gauss, was long in close touch with him. Making every possible allowance for his partiality, the following quotations from his biography are valuable as giving a substantially accurate estimate of Gauss's character:—

"Gauss as the son of poor but upright parents was from his youth accustomed neither to the luxuries nor to the refinements of our times. The means, just short of being wretched, which were available for his existence in the first years of his life, were enough for his requirements. The little that was his share he knew from his earliest days to use with wise economy in such a manner that to him always a small excess remained for use in unexpected circumstances, and protected him from being in a position in which he would feel compelled to depend on the material support of his fellow-men. High personal self-respect as well as the thought of preserving his independence of mind, undoubtedly brought him to this mental disposition to which he remained faithful with iron consistency from his youth up to the last days of his life*." As an illustration, may be mentioned his behaviour when in the first few months of his Professorship at Göttingen, even before he had drawn his small salary, a war contribution of 2000 francs was imposed on him by the French regime. It was impossible for Gauss to pay this sum. So, Olbers sent it to him with expressions of regret that such a savant as Gauss should suffer this penalty. But Gauss sent back the sum to Olbers. A few days later Laplace wrote to Gauss that the sum had been deposited (by Laplace) on his behalf in the Paris Treasury. But Gauss refused the gift and sent to Laplace in a few days the whole amount together with the interest. Again, if Gauss had shown the least wish to approach Napoleon for any favour, it would have surely been accorded to him, because of Napoleon's known love for Mathematics. But Gauss never bent his knees to the French, although at the time nobody could dare to think that the region from the Elbe to the Niemen would be ever freed from the French. At this time the suggestion was made by his publisher, Perthes, that Gauss should write his *Theoria Motus* in French, but the book was written in Latin instead.

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* Joseph Gauss, who was first artillery officer in Hanoverian service and then Ober-baurat in Hanoverian railway directorate.

† Eugen Gauss (1811—1896) and Wilhelm Gauss (1813—1879), both of whom emigrated to America and have left there a large number of descendants. Eugen began as a merchant in Missouri and then settled down as a farmer near Columbia. Wilhelm was first a farmer, then merchant, again farmer and died as a whole-sale shoe merchant at St. Louis.

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* V. Walterhausen, l. c, pp. 77-78.

Von Walterhausen further says: "If Gauss had wished to carry out ambitious (non-scholarly) plans in life, he would have been able to do so by means of his genius. But he had never stretched his hands even for the honours which people brought to his doors; he remained up to his last days, as he was in his youth and in his hoary years, the simple Gauss. A small study room, a small working table with a green cover, a writing desk (for standing at) painted with white oil colour, a narrow sofa and an armchair after his 70th year, a single dark-burning light, a bed-room that could not be heated, simple food, a sleeping coat and a silk cap, these were all his necessities.........The unshatterable thought of the continuity of one's personality after death, the firm belief in a last Ordainer of things, in an eternal just, all-knowing, almighty God, formed the foundation of his religious life."*

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* V. Walterhausen, l. c., pp. 102–103.

"Gauss was for us the sacred example of a nature true through and through, in his mental life as well as in his emotional life. All glitter was averse to his soul; he punished with sovereign contempt and with bitter irony all charlatanisms, specially of a scientific nature." "The man, he told me once", says v. Walterhausen "who has after knowing his mistake persisted in it is to me most contemptible."

As regards his interest in politics, v. Walterhausen says that Gauss regularly visited the literary museum and read eagerly the newspapers of all nations, from the English *Times* to the Göttingen *Weekly-sheet.* Gauss was of a thoroughly aristrocratic and conser-native nature and to him an absolute Government conducted with high intelligence was preferable to all other forms of Government. He used to say *Mundus vult decipi* and looked upon agitators with distrust.

CHAPTER II.

CAUCHY

Augustin—Louis Cauchy was born at Paris on the 21st of August, 1789, in a pious family which had the good fortune to belong to that middle class of society which is exposed neither to the sufferings due to poverty nor to the dangers due to riches. His father, Louis-Francois, born at Rouen in 1760, was first an advocate in the Parliament of Normandy, then joined Government service, came to Paris in 1785, and was Secretary—Archivist to the Senate from about 1800 or 1801 and to the Chamber of Peers from 1814 to 1830. Thus Cauchy's infancy and youth were lived entirely at Paris. Of two brothers, both younger than himself, one became a member of the highest court of justice and the other succeeded his father as Secretary—Archivist to the Chamber of Peers.

Cauchy's classical education was commenced early at home under his father and it was continued later under able teachers at the E'cole centrale du Pantheon. He left that school in 1804 at the age of 15 years "after two years of rhetoric, carrying with him, as a result of

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the general competetion, the second prize for Latin discourse, the first prize for Greek version and the first prize for Latin verse. This universality of success was recognized by the Institute by the award of the wreath reserved for the student, who of all the students of the central schools distinguishes himself in Humanities."* The classical education of Cauchy was the result of Lagrange's advice given to Cauchy's father in the following circumstances.† Lagrange was a member of the Senate where Cauchy's father was the Secretary—Archivist. The son was presented to Lagrange on many occasions, and the great mathematician, having noticed the precocity of the child, spoke to the father as follows: "Do not let him open a book on Mathematics or write a symbol, until he has finished his literary studies."

"After having spent a year in attending the public lectures on Mathematics by an excellent teacher, Dinet, the young Cauchy found himself in a fit state to appear at the examination for admission to the E'cole Polytechnique. He secured the second place in order of merit among the successful candidates of 1805 when he was 16 years of age. At the end of his two years' course, he left the E'cole in 1807 as the third in order of merit of that year. It will be noted that Cauchy's scientific apprenticeship was neither retarded nor expedited by his having been prepared by the study of letters in accordance with the advice of Lagrange."*

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* See pp. 143—160 of the *Melanges scientifiques et littéraires,* Vol. 3, (Paris, 1858) of Jean-Baptiste Biot (1774—1862); see specially p. 144.

† See Biot's book, *l. c.*, p. 144.

At this stage it will be desirable to give an idea of the E'cole Polytechnique† as it was in the first decade of the 19th century. The school, established in 1794 during the revolutionary troubles, was a unique institution‡ in Europe. It was primarily intended to train would-be officers for the revolutionary forces and later for the army of Napoleon. Its course covered two years and admission to it was by an annual competetive examination as a result of which the first 150 in order of merit were taken. The position of an examinee depended entirely on the record of the marks obtained in the examination. The total marks in each subject aggregated to 2000 and of these over 90% could be actually secured as shown by concrete cases§: this will give an idea of the keenness of the competetion.

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* *Ibid*, pp. 144—145.

† See Klein's *Entwicklung*, *l. c.*, pp. 63-66, from which pages the following account is taken.

‡ The prevalent ideal of education in the Europe of the 18th century was an all-sided harmonious building up of personality. As a contrast to this, the ideal set in the E'cole Polytechnique was essentially practical. The product of the school was to be of the greatest practical service to the State.

§ For example, according to Klein, in the last quarter of the 19th century, J. Hadamard, the famous mathematician, secured as many as 1875 marks, These two memoirs appeared in the *Journal de l' E'cole Polytechnique* in 1813 and were the first publications of Cauchy. Writing about these memoirs, Legendre says that the young author has succeeded in overcoming a difficulty which had resisted the efforts of master mathematicians and that the solution of the difficulty has led to the perfection and completion of the theory of solids. With these publications, Cauchy became a marked man who was expected to achieve in course of time the highest success in the field of mathematical research.

The third publication of Cauchy appeared in the same volume of the *Journal de l' E'cole Polytechnique* as the two mentioned above. It is entitled "Researches on Numbers"* and its genesis is described by Cauchy as follows :—"M. Lagrange has demonstrated for the first time in the *Memoirs of Berlin* for the year 1770 the following theorem : Being given a prime number p and two other integers, B and C, positive or negative but not divisible by p, one can always find two integers t and u, such that the expression

$$t^2+Bu^2+C$$

is divisible by p. This theorem has analogy with another simpler of which the enunciation is the following : Being given a prime number p and two integers, A and B, positive or negative but not divisible by p, one can always find an integer x such that the expression $Ax+B$ is divisible by p. This last theorem having been proved very simply by M. Legendre in his *Introduction to the theory of numbers*, the analogy led me to believe that there must exist a similar proof of the theorem of M. Lagrange. The work, which I undertook with the intention of finding this proof, has led me also to the proofs of some other theorems which I have expounded one after the other and of which many appear to be new."

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* See *Oeuvres*, s. 2, t. 1, pp. 39–63.

After giving an idea of the first three publications of Cauchy, we proceed to mention briefly his other writings before his admission to the membership of the Institute, reserving for about the end a detailed account of the more important of them.

The famous work, "Memoir on definite integrals"* of about 180 pages was read before the Institute on the 22nd August, 1814, although it was published by the Academy of sciences as late as 1827. In this important work, Cauchy gave, though not very explicitly, the famous integral—theorem which is the foundation of his theory of functions of a complex variable. In 1815, there appeared in the *Journal de l' Ecole Polytechnique* two memoirs,† one, "Memoir on the number of values which a function can acquire when one permutes in every possible manner the quantities contained in it", and a larger work, "Memoir on the functions which can take only two values, equal and of opposite signs, by consequence of the transposition of the variables which are contained in the functions", which was read before the Institute on the 30th November, 1812. These two memoirs are said to have inspired Abel 12 years later with his proof of the impossibility of the general solution of the quintic by means of radicals alone. In the same volume of the *Journal* was printed the "Memoir on the determination of the number of the real roots in algebraic equations"* (read at the Institute in the year 1813) which gave, before Sturm gave his theorem, a sure method for finding the number of real roots between two given limits. Next may be mentioned the "Memoir on the system of values which it is necessary to attribute to different elements determined by a large number of observations, in order that the greatest of all the errors, abstraction being made of the sign, should become a minimum",† which was presented to the Institute on the 28th February, 1814. In this series may be mentioned the famous work,‡ "Theory of the propagation of waves on the surface of a fluid having an indefinitely great depth" of over 300 pages (in its final form), the manuscript of which was sent to the Institute for the competetion of 1815 for the Grand Prize in Mathematics which was awarded to Cauchy for this work. Another interesting memoir* is "Demonstration of the general theorem of Fermat on the polygonal numbers" (read before the Institute in 1815) in which he proved and extended the following theorem enunciated by Fermat: Every integral number can be formed by the addition of three triangular numbers, of four squares, of five pentagonal numbers, of six hexagonal numbers and so on; only some particular cases of which had been proved by Legendre and Gauss.

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* See *Oeuvres*, s. 1, t. 1, pp. 319—506.

† See *Oeuvres*, s. 2, t. 1, pp. 64—90 and pp. 91—169.

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* See *Oeuvres*, s. 2, t. 1, pp. 170—257.

† See *Oeuvres*, s. 2, t. 1, pp. 358—403.

‡ See *Oeuvres*, s. 1, t. 1, pp. 1—318.

Cauchy's reputation at this time (about 1816) had risen so high that he was sure of getting into the mathematical section of the Institute in the first vacancy. But his admission actually took place under unfortunate circumstances which may be briefly described here. After the crisis of a hundred days during which Napoleon had returned to France from Elba, a royal ordinance, dated the 21st March, 1816, was promulgated by which the old Academies under their original names, viz the French Academy, the Academy of Sciences, the Academy of Inscriptions and Belle Lettres, and the Academy of the Fine Arts, were re-established. The same ordinance also gave the list of the members of each of the restored Academies, and in the list for the Academy of Sciences, the names of Breguet and Cauchy were found substituted for the names of Carnot and Monge (1746–1818). According to Biot, the exclusion of Monge was a "political inhumanity" towards him; for, Monge had never had a seat in the revolutionary assemblies. "Feeble and defenceless, like many others he had bent his head before the storm of the Terror, which, however, unfortunately for him, had carried him to the Ministership of Marine." The fact that he had trained 2000 pupils for scientific, civil and military careers in France, and the devotion which he had always displayed towards his colleagues and pupils at the E′cole Polytechnique, were all forgotten. Cauchy had never been a pupil of Monge, and "his personal feelings did not allow him to conceive any doubt about the legality of his nomination and he accepted it without hesitation".* "The opinion of the scientific world was indulgent towards Breguet but severe towards Cauchy."†

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* See *Oeuvres*, s. 2., t. 6, pp. 320–353.

Towards the end of the year 1815, Cauchy was nominated Associate Professor of Analysis at the E′cole Polytechnique and became titular Professor in 1816. Later, he occupied a chair in the Faculty of Sciences of the Paris University, and was also *suppleant du cours* of Mathematical Physics at the College de France; these two appointments he held in addition to his post at the E′cole Polytechnique. Cauchy was above everything a man of duty, and

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* Biot, *l. c.*, pp. 147–148.

† *Ibid*, p. 148

"called to teach he turned all his thoughts to teaching." At the E'cole Polytechnique he was very popular with his pupils, one of whom, Combes, later Member of the Institute, spoke* of him as follows in 1857: "In M. Cauchy, allied to the genius of Euler, Lagrange, Laplace, Gauss, Jacobi, there were the love of teaching amounting to enthusiasm, a rare goodness, a simplicity and a warmth of heart which he retained up to the end of his life." The lectures which he delivered at the E'cole before 1820, he began in 1821 to publish in the shape of books. These books are the famous works,

Cours d' Analyse algebrique†, 1821 ;

Resumé des lecons données sur le calcul infinitesimal,‡ 1823 ;

Lecons sur les applications du calcul infinitesimal â la geometrie§, 2 volumes, 1826—1828

Lecons sur les calcul differentiel ||

In the year 1826, Cauchy started a periodical of his own, called "*Exercises des Mathematiques*¶, the

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* *Ibid*, p. 157

† *Oeuvres*, s. 2., t. 3.

‡ *Oeuvres*, s. 2, t. 4, pp. 1–261.

§ Vol. 1 in *Oeuvres*, s. 2, t. 5; Vol. 2 in *Oeuvres*, s. 2, t. 6.

|| *Oeuvres*, s. 2, t. 4, pp. 265–615.

¶ It was called *Exercises des Mathematiques* from 1826–1830 ; bore the same title from 1830 to 1835 but was published at Turin and the at Prague. Later its title was *Exercises de Analysis Mathematiqe et de Physique.*

advertisement of which gives the following : "This work shall consist of a series of articles on the different branches of mathematical science. It shall appear in parts which shall follow one another at short intervals. In those articles, it is proposed to pass under review the different branches of Analysis, to elucidate the difficulties which present themselves and to offer new methods with the help of which one may treat most easily questions already solved or solve those which have remained unsolved. The principal applications of those methods will be to Physics, to Mechanics and to the Theory of Numbers". The first volume contains 30 articles and is in over 400 pages. Cauchy continued the publication of his *Exercises* up to the time of his death. Abel had a high opinion of the value of that periodical and wrote to a friend after reading an issue that "of all mathematicians, Cauchy is really the one who understands best how Mathematics ought to be studied". According to Biot* "the new methods and new views, scattered in the issues of the *Exercises*, have been not only for the author but also for many other mathematicians the fertile initiatives for a multitude of brilliant results."

With the revolution of 1830, which placed Louis Phillipe of Orleans on the throne, began a new period in Cauchy's life. At that time, Cauchy was a married man with two daughters, and was holding, in addition to his post at the École Polytechnique, an associate Professorship in the Faculty of sciences at the Paris University and also an assistant Professorship of Mathematical Physics at the College de France (*as suppleant du cours*). The peaceful existence which he had been leading was disturbed by the demand of the new Government for an oath of allegiance from all public functionaries. Now, Cauchy's position was well-known. "Without having ever solicited or received any personal favour, Cauchy venerated the family of Charles X as the legitimate heirs to the royalty, as the supporters of religion, as the sole and assured refuge for France after so much of troubles and errors. He loved the members of that family also for what they had suffered. He thought that he was required (by the new Government) to perjure himself, and, abandoning all his posts and his family, he sought refuge in Switzerland."* As soon as the King of Sardinia came to know of Cauchy's voluntary exile, he created for Cauchy a special Professorship of Mathematics at the University of Turin. Cauchy occupied this chair with pleasure and it is interesting to note his linguistic capacity in the fact that he lectured at Turin in Italian.

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* l. c , p. 149.

In the year 1832, Cauchy resigned the chair at Turin, having been called to Prague by Charles X for taking part in the education of his grandson, Comte de Chambord. Cauchy sent for his wife and two daughters and followed with them the princes to Goritz. During the six years of the honourable connection with the royal family, Cauchy found time to write a large number of precious memoirs. Among these may be mentioned here his "Memoir on the dispersion of light" (Prague, 1836).

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* Biot, *l. c.*, p. 150.

In 1838, Cauchy returned to France bringing with him the affection and esteem of the royal family, and took his seat in the Institute, which, contrary to practice, had been left vacant, no doubt in admiration of Cauchy's genius. From 1838 to 1848, Cauchy was without any routine duty of any post. In 1840, when the seat of Poisson on the *Bureau des longitudes* fell vacant by Poisson's death, Cauchy was elected by the Bureau to Poisson's place. But the oath of allegiance was obligatory in the case of every member of the Bureau and it was clear to all that Cauchy would not agree to take the oath. Lengthy negotiations were carried on by friends on both the sides—Cauchy's and the Government's—to reduce the formality to as simple terms as possible. But all efforts proved to be futile and Cauchy's election was not confirmed. The revolution of 1848 brought about a republican Government which did not demand any oath of allegiance from savants, and so Cauchy came to occupy his chair of Mathematics at the Paris University, the only one of his former posts which was at the time vacant. When the Government of Napoleon III was established in 1851 and required an oath of allegiance, Cauchy retired from his Professorship. But later on, the effort of the Minister of Public Instruction, Fortoul, succeeded in securing from the Emperor, the authorization to send Cauchy back to his chair without any condition whatsoever. Cauchy showed his appreciation of the Emperor's conduct by devoting the whole of the income derived from the University to charitable purposes in the little commune of Sceaux where he resided.

An idea of Cauchy's productivity may be given here. The total number of his publications have been estimated by his pupil and biographer, C. Valson, to be 789, including eight books complete in themselves. Of this total, about 500 memoirs were published during the 19 years from 1838 up to his death in 1857. The ten or twelve memoirs, which he produced before 1816, have been already mentioned. During the period 1816—1830, he produced his books and over 200 memoirs. Cauchy's publications are on a wide a range of subjects, viz—theory of numbers; theory of algebraic and transcendental equations; Galois groups; algebraic analysis; combinatorial analysis; differential and integral calculus; definite integrals, specially Eulerian integrals; theory of functions of a real variable; theory of functions of a complex variable; elliptic functions and their applications; hyperelliptic, Abelian and Fuchsian functions; differential equations; functional equation; interpolation and difference equations; calculus of variations; theory of groups of transformations; theory of aggregates; geometry of polyhedra; analytic geometry; including differential geometry; mechanics; capillarity; light; heat; electricity; astronomy; celestial mechanics; geodesy; errors of observations and calculus of probabilities.

I proceed now to consider in detail each of the important publications of Cauchy; a list of such publications is given at the end of this chapter. I shall begin with Cauchy's first great book, viz. *Cours d' Analyse algébrique*

(I)

This book which covers the whole of Cauchy's *Oeuvres*, s. 2, t. 3, and comes to 484 pages, including the introduction and the nine notes, was ushered into existence with the following words of Cauchy: "Some persons who have been very pleased to guide my first steps in the career of science and among these I cite with recognition Messrs. Laplace and Poisson, having indicated the desire to see me publish the course of analysis of the royal polytechnic school, I have decided to reduce that course to writing for the greatest utility of the pupils. I offer here the first part, known under the name of *Algebraic Analysis*, and in which I treat successively different species of functions real or imaginary, of series convergent or divergent, of the solution of equations, and of the decomposition of rational fractions. In speaking of the continuity of functions, I have not dispensed with making known the principal properties of infinitely small quantities, properties which serve as the foundations of infinitesimal calculus. Finally, in the preliminaries and in some notes placed at the end of the volume, I have presented developments which can be useful be it to the professors and students of the royal colleges, be it to those who wish to make a special study of Analysis."

That the book revolutionized the teaching of Analysis, not only in France, but throughout Europe, will be easily understood when we mention the following features (among others) which were quickly introduced in English, German and Italian books on the subject. *First*, the notion of the continuity of a function which prevailed at the time, as handed down in the writings of Euler and Lagrange, was given up and the definition, now in common use, was introduced. According to Euler and Lagrange, "a function was called continuous or discontinuous according as the different values of it corresponding to different values of the variable are, or are not, subject to one and the same law, are or are not furnished by one and the same equation."* This definition would, for example, lead to the absurdity that a function, defined to be equal to x for positive values of x and to $-x$ for negative values of x, would be discontinuous, but the same function, considered as given by

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* See Cauchy's "Memoire sur les fonctions continues" (Comptes Rendus, t. 18).

$$\frac{2}{\pi}\int_{0}^{\infty}\frac{x^2\,dt}{t^2+x^2},$$

will be continuous. In fact the vague and indeterminate character of continuity as defined by Euler and Lagrange was first clearly made known to the mathematical world by Cauchy. The *second* feature of Cauchy's book is the introduction, for the first time, of complex quantites, in all the operations to which real quantities had till then been submitted. Thus the exact significance of

$$a+b,\ a-b,\ ab,\ \frac{a}{b},\ a^b,\ \log_b a$$

for all the possible values, real or complex, of a and b, was given; and also the functions,

$$A^x,\ \sin x,\ \cos x,\ \log x,\ \sin^{-1} x,\ \cos^{-1} x,$$

were clearly defined for complex x. The *third* feature of Cauchy's book may be stated to be his treatment of infinite series. Starting with a clear definition of the meaning of the sum of an infinite series, Cauchy gives many numerical examples and shows how the remainders can be approximated to. Klein is quite right in saying*: "How enormous is Cauchy's service in the exposition of these things, already known to a greater extent, admits of being measured first by a comparison with his predecessors and contemporaries. The book differentiates itself on the side of principles as much from most of the indefinite attempts, which had been dominant up to that time, to found Infinitesimal Calculus on perception, as from the entirely external formal point of view adopted, in his *Theorie des fonctions analytiques* and in the *Lecons sur les calcul des fonctions*, by Lagrange who seeks to disclose the kernel of the new world of thought. The book gives, instead of these, an arithmetical foundation free from objections in regard to all critical points: with this fundamental work begins the so-called "arithmetization of the whole of Mathematics".

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* See *Entwicklung*, *l. c.* pp. 83–84

Before we conclude, it is necessary to warn the reader that Cauchy is not *always* right. For example, on p. 120, he enunciates the inaccurate theorem that a convergent series of continuous functions is necessarily continuous in the interval of convergence, and gives a false proof of it by avoiding the important notion of uniformity of convergence.

(II)

Résumé des lecons donnees a l'école royale polytechnique sur le calcul infinitésimal (1823).

In the advertisement with which the book begins we read the following words of Cauchy : "This work, undertaken on the demand of the council of instruction of the royal polytechnic school, offers the summary of the lectures which I have given at this school on the infinitesimal calculus. It shall be composed of two volumes corresponding to the two years which form the period of instruction. I publish to-day the first volume, divided into 40 lectures, of which the first twenty are on the differential calculus and the last twenty on a part of the integral calculus. The methods which I have followed differ in many respects from those which will be found expounded in the other works of the same class. My principal aim has been to reconcile rigour, which I have made a law to myself in my *Cours d' Analyse*, with the simplicity which results from the direct consideration of infinitely small quantities. For this reason I have believed it to be my duty to reject the developments in infinite series, each time that the series obtained are not convergent; and I have found myself forced to relegate to the integral calculus the formula of Taylor, this formula no more being admitted as general except when the series which it includes is found reduced to a finite number of terms and completed by a definite integral. I do not forget that the illustrious author of *Mecanique analytique* has taken the formula in question as the base of his theory of *fonctions dérivées*. But, in spite of all the respect which such a great authority commands, most of the mathematicians agree now to recognize the uncertainty of the results to which one may be led by the employment of divergent series, and we add that in many cases, the theorem of Taylor appears to furnish the development of a function in convergent series, although the sum of the series differs essentially from the proposed function [; see the end of the 38th lecture]. Besides, those who will read my book, will convince themselves, I hope, that the principles of the differential calculus, and its most important applications, may be easily expounded without the intervention of series.

In the integral calculus, it has appeared to me necessary to demonstrate generally the existence of the *integrals* or primitive functions before making known their diverse properties. For attaining this object, it has been found necessary to establish at the outset the notion of the *integrals taken between given limits* or *definite integrals*". . . . .

Writing about this book, Klein says; "It treats of the questions of the fundamental principles of Infinitesimal calculus. It contains the rigorous exposition of this subject on the basis of the notion of limit and freed from all Metaphysics; the exposition which has since become standard. In contradistinction with the Cours d' Analyse, the book moves almost throughout in the real region. The corner-stone of the whole edifice is made the mean-value theorem, known since a long time, and in any case since Lagrange, and which we write in the modern style of expression introduced by Cauchy, as

$$\frac{f(x+h)-f(x)}{h} = f'(x+\theta h).$$

The integral calculus begins on p. 122 with the definition of the definite integral for which is given there the arithmetical existence-proof. First, attaching to that is given on p. 214 the development of functions in Taylor's series . . .

. . . .

The theorem [fundamental theorem of infinitesimal calculus, viz.

$$\frac{d}{dx}\int_{0}^{x} f(x)\,dx = f(x) \Big]$$

is not found here and is first comprehensively enunciated in the book of Abbe Moigno which he brought out in 1840—44 according to Cauchy's ideas and at Cauchy's suggestion."

(III)

"*Theorie de la propagation des ondes a la surface d' un fluide pesant d'une profondeur indéfinie*" (1827).

This work which received the prize announced by the Institute of France for solving the following problem: "A fluid initially at rest and of an indefinite depth has been set in motion by the effect of a given cause. At the end of a given time are required the form of the external surface of the fluid and the velocity of any of the molecules situated on that surface", covers pp. 1—318 of *Oeuvres*, s. 1, t. 1, and contains on pp. 8—113 the memoir proper and on pp. 115—318 the 20 notes of which the first 13 were sent up along with the memoir in 1815 and the remaining 7 were added by Cauchy later on.

"For the sake of convenience, I shall divide the solution, that is to say, the memoir which contains it, into three parts. In the first part, I shall show how one can deduce the equations which express the initial state of the fluid, when one knows at the beginning the form of the external surface and the forces which act on it. I shall give in the second part the equations which determine, at any time whatever of the motion, the state of the fluid mass and that of its surface. Finally, in the third part, I shall establish the general laws which result from the formulae given in the second, and I shall determine the numerical values of the constants which enter in the expression of these laws. For greater convenience, I relegate to the end of the memoir in separate notes the proofs of different analytical formulae which I have used for the solution of the problem".*

(IV)

"*Mémoir sur les intégrales définies*" (1827)

This memoir which appears in *Oeuvres*, s. 1, t. 1, with two supplements, and a number of footnotes which were added by Cauchy "for indicating the manner of simplifying the formulae or for deducing new results," contains the germ of Cauchy's integral-theorem; still he was not aware of the fact that he was laying the foundation of a theory of functions of a complex variable, as is clear from (1) the following words of the introduction and (2) the report of Lacroix and Legendre to the Institute :—

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* See Cauchy's preface on p. 7 of *Oeuvres*, s. 1, t. 1.

(1) Cauchy says: "The solution of a large number of problems reduces itself in the last analysis to the evaluation of definite integrals; accordingly, mathematicians have occupied themselves much with their determination, One finds in this connection a multitude of theorems, curious and useful, in the memoirs and *Calcul Intégral* of Euler, in many memoirs of M. Laplace, in his *Recherches sur les approximations de certaines formules*, and in the *Exercises de Calcul intégral* of Legendre. But, among the different integrals obtained by the first two mathematicians which I have cited, many have been discovered for the first time by the aid of a kind of induction based on the passage from the real to the imaginary. The passages of this nature often lead in a very prompt manner to results worthy of remark, Nevertheless this part of the theory is, as M, Laplace has observed, subject to many difficulties. Accordingly, after having shown in the calculus of generating functions the resources which Analysis can derive from similar considerations, the author adds: 'one can therefore consider these passages as the means of discovery similar to the induction of which the mathematicians have for a long

time made use. But these means, although employed with a good deal of precaution and reserve, leave always to be desired proofs of their results. For removing this inconvenience the author has taken pains to confirm by other methods the values of the integrals which he has found."

(2) Lacroix and Legendre conclude their report as follows: "We do not examine if the new methods of M. Cauchy are simpler than those already known, if their application is easier, and if one can find by their help some result which the known methods cannot give; for, supposing that one were to answer these questions in the negative, the merit of the author will not be less,

(i) for having constructed, by a uniform course, a series of general formulae suitable for transforming definite integrals and for facilitating their determination;

(ii) for having been the first to remark that a double integral, taken between given limits for each variable, does not always give the same result in the two ways of effecting the integrations;

(iii) for having determined the cause of this difference and for having given the exact measure, by the help of *singular integrals*, of which the idea belongs to the author and which may be regarded as a discovery in Analysis;

(iv) finally for having given, by his methods, new integral formulae, very remarkable, which may well be deduced by the known methods but which no person had as yet found.

It appears to us, for all these reasons, that M. Cauchy has given in his researches on the definite integrals a new proof of his sagacity which he had shown in many of his other productions; we think, therefore, that his memoir is worthy of the approval of the class (of mathematics) and to be printed in the *Recueil des Savants étrangers.*"

The reason why the integral-theorem

$$\int_C f(z)dz = 0$$

is recognized as given in this paper is this:—

Denoting

$$f(x \pm iy) \text{ by } P' \pm i P'',$$

$$f(x) \text{ by } p,$$

$$f(\pm iy) \text{ by } p' \pm ip'',$$

Cauchy deduces the two equations[*]

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[*] *Oeuvres* s. 1, t. 1, p. 340.

$$(5)\quad\begin{cases}\displaystyle\int_0^x P'dx-\int_0^x p\,dx=\int_0^y p''dy-\int_0^y P''dy\\[2ex]\displaystyle\int_0^x P''dx=\int_0^y P'dy-\int_0^y p'\,dy\end{cases}$$

which, when the memoir was being printed in 1825, he recognized* to be equivalent to the single equation

$$\int_0^x f(x+iy)\,dx-\int_0^x f(x)\,dx$$

$$=i\left\{\int_0^x f(x+iy)\,dy-\int_0^y f(iy)\,dy\right\}$$

which is

$$\int_C f(z)\,dz=0,$$

C being the rectangle of sides x, y with a vertex at (0,0).

For many years, no body recognized the above theorem as the equivalent of the equations (5). Very probably this was because (1) Cauchy used neither figures nor geometrical language, (2) he refrained from using complex quantities, always carefully separating an equation into its real and imaginary parts.*

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* *l. c.*, foot-note, equation (B).

(V)

Memoirs on differential equations†.

About these the following remarks of Klein‡ deserve to be quoted: "The subject of of differential equations has been worked out by Cauchy in so many directions that it appears absolutely impossible to mention all his results or even to give a survey of the store of his publications belonging to the subject. I content myself, consequently, with bringing forward some essential points.

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* The statement, first made by Gauss in 1811, that $\int_{z_0}^{z_1} f(z)\,dz$ has the same value even if the path from z_0 to z_1 is changed, was first proved by Cauchy in his "Memoire sur les integrales définies, prises entre des limites imaginaires", which appeared in 1825 as a booklet and again in the *Bulletin des Sciences Mathématiques* of 1874 and 1875 in two instalments. It appears also in Ostwald's *Klassikern.*

The memoir consists of 18 articles and an addendum. Art. 1 is the introduction and contains historical remarks. Art. 2 gives the definition of a definite integral between imaginary limits. Art. 3 contains a proof of Gauss's statement based on the Calculus of Variations. Art. 4 gives the residue theorem. After considering some general theorems, Cauchy applies in Arts. 12-18 his results to evaluate a large number of definite integrals and to sum a number of series; the addendum gives a collection of all the definite integrals evaluated in this manner.

† See e. g. *Oeuvres*, s. 1, t. 2, pp. 5-8; s. 2, t. 2; s. 2, t. 12, pp. 272-309; s. 1, t. 7, pp. 5—17.

‡ *Entwicklung*, l. c., pp. 85-86.

"Also in this subject is due to Cauchy the fame as the first person to have given the existence-proof for the solutions of any differential problem in non-singular region for which solutions any initial values are suitable. Of the different methods which Cauchy applies, I mention here only the two best known:—

1. The differential problem is replaced by a difference problem—the sought function curve by a polygon—and from its solution by diminishing the intervals giving the sides of the polygon the limit is taken. By showing the convergence of the process to a limit—just the solution of the differential problem—the existence-proof of the latter is given. This process corresponds exactly to the definition, given by Cauchy, of the definite integral and its numerical calculation (for practical approximations improved into Simpson's rule).

2. One assumes that the coefficients of the differential equation admit of being expanded in convergent power series. Then for the sought integrals can be formally laid down power series whose convergence can be proved by the setting up of majorants. Cauchy called this proof—process the 'methode des limites' and extended it also to the complex region.

"Also these thoughts of Cauchy owe their origin already to the twenties. We find in them step by step the beginnings of the arithmetized modern Analysis. More wonderful must it seem to us, that exactly these works have come out of the teaching activity at the polytechnic school, an evidence of the extraordinarily high requirements on the purely mathematical side on which the culture directed towards practical side was founded.''

(VI)

*Résumés Analytiques**

Cauchy says in the advertisement: "I have thought that a series of articles destined to offer the summaries of the most important theories of Analysis, be they old or new, particularly of the theories which embrace algebraic Analysis and of methods which render the exposition easier, may interest mathematicians and those who are devoted to the cultivation of the sciences. Such is the aim which I propose to myself in the present work which shall appear by parts." There are 22 articles on topics which will be considered to-day to be elementary. As examples, may be mentioned: the first article which is on figurate numbers; the last article which is on the expansions of log $(1+x)$ and $(1+x)^{\mu}$ when x is imaginary.

(VII)

Memoirs on the theory of Elasticity and Optics†. "The differential equations of the elasticity of three-dimensional bodies were first laid down in 1821 by Navier, and indeed from interest in Technics, starting from a perception based on the molecular theory, and these equations were given for the case of 'isotropic' media alone. Cauchy founds from about 1825 onwards a phenomenological mode of exposition of the relations, in which the body is regarded as a continuum, and which now makes use of the two 'tensors', *strain* and *deformation*. This notion-building signified a great progress as against the notion of a fluid pressure, continued uniformly towards all sides, as the only one known to the 18th century. In 1827 Cauchy extended the treatment to anisotropic (crystalline) media and starting with these fundamentals, from 1828-30 he succeeded in giving the mathematical foundation of the simple theorems of the Fresnelian Optics, an achievement which from the beginning of his labours he had placed before himself as one of his chief aims. Frankly, his theory shows in two important points a deviation from the observations and conceptions of Fresnel :—(1) it requires that the oscillations of polarized light which according to Fresnel lie in the plane of polarization ought to be perpendicular to that plane ; (2) it furnishes in all media, along with the transversal vibrations, also longitudinal vibrations which according to observations play no part in Optics. These two controversial questions have for decades engaged the heads (of scientists) until the final supremacy of the Maxwellian theory. Indeed even in 1896 by the discovery of the Röntgen rays the thought was public that they were concerned with the long sought optical longitudinal vibrations.

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* Turin, 1833, 182 pages. See also *Oeuvres*, s. 2, t. 10.

† See *Exercises*, 1827, 1828. Also *Bulletin Philomatique*, 1823, pp. 9—13.

"Besides these discrepancies remained now, however, still an entire range of optical phenomena which did not at all admit of being grasped with the phenomenological theory of Optics: *the dispersion of light*. In order to explain that, Cauchy went back in 1835 or 1836 in his *Mémoire sur la dispersion de la lumiere** (Prague) to molecular perceptions and arrived at his object at least qualitatively by means of the supposition, that the molecular distances in comparison with the wave-lengths of light are not vanishingly small. Even to-day Cauchy's formula of dispersion: $n=a+\frac{b}{\lambda}+\frac{c}{\lambda^2}+\ldots$ is applied for media whose absorption-bands lie in the ultra-reds"†. Klein points out that the investigations of Cauchy on mathematical physics along with those of Fourier and Poisson took root early in German Universities and became quickly international property.

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* The memoir was published by the Royal Society of Sciences of Prague and as a part of the *Exercises de Mathematique*. The following quotation from the preface by Cauchy is of interest as reflecting the author's temperament: "One knows well what circumstances made it a duty for me to renounce the three chairs which I was occupying in France and what august wishes made me give up later the chair of Mathematical Physics which the King of Sardinia had deigned to give me. But it is not without doubt that in the presence of the descendants of Lois XIV, in the presence of those princes, the protectors of letters and sciences, I can myself agree to dispense with my continued efforts for contributing to their progress. The new exercises shall appear as the preceding, in parts and if possible—for on this earth and in this century one can give no undertaking for to-morrow—they will follow one another at short intervals. The first part deals wholly with the memoir on the dispersion of light, a memoir of which the first 2 paragraphs only were published in 1830."

† *Entwicklung, l. c.*, pp. 73–74.

(VIII)

Mémoire sur la théorie des nombres.

The memoir proper together with 14 notes added after its presentation to the Academy of Sciences on the 31st May, 1830, covers the whole of Vol. 3 of *Oeuvres*, series 1, and comes to 452 pages. The printing of the memoir was delayed, as Cauchy says, on account of his prolonged absence from France for 8 years. The memoir proper of 78 pages is divided into three parts, called paragraphs by Cauchy ; the headings of the 2nd, 3rd and 4th parts are "new applications of the formulae established in the first paragraph, continuation of the same subject, continuation of the same subject." Among the notes are the 4th dealing with quadratic residues, the 6th on the sum of the primitive roots of a binomial equation and on the symmetric functions of these roots, the 13th on the quadratic forms of certain powers of prime numbers or of the quadruple of those powers, the 14th on observations relating to the quadratic forms under which present themselves certain powers of prime numbers and the reduction of the exponents of those powers.

From what has been said in the preceding pages of this chapter, an estimate of Cauchy's work can be made without difficulty. He was universally admitted even in 1830 to be one of the greatest mathematicians of the world and after 1850 to be second only to Gauss. He had not only advanced the bounds of mathematical knowledge in numerous directions but had also revolutionized the teaching of Higher Mathematics by banning the use of divergent series and by building up algebraic analysis on a sound basis. He was the founder of the differential and integral calculus as it is taught at the present time on the basis of limit, he was also the founder of the theory of functions of a complex variable to which he gave not only the fundamental integral-theorem but also the theorem called Cauchy-Taylor theorem, the notion of the circle of convergence and Laurent's theorem. He perfected the geometry of solids by his classical researches on polyhedra ; he was the first to discuss and prove the existence-theorem in the theory of ordinary differential equations ; he gave a new method for solving non-linear partial differential equations ; he enriched the theory of numbers by giving new theorems and by giving a new proof of the reciprocity theorem. He published extensive investigations on the transformation of the first variation of multiple integrals. In the fields of Astronomy and Mathematical Physics, Cauchy published valuable researches; he invented an analytical method by which all the inequalities of long periods of planets could be directly determined ; he enriched Hydrodynamics by giving the theorem about the permanence of the irrotational motion of a liquid and by his investigations on surface waves. He built the modern foundation of the theory of elasticity by treating the body as a continuum and by operating with the notions of stress and strain; he extended his treatment to crystalline media and deduced the mathematical foundations of Fresnel's theorems.

But all these achievements of Cauchy would have left to posterity a higher reputation for him, had he not suffered from an inordinate desire to communicate his results and to publish them. Even Biot* notes this defect. "During the last 19 years of his life, he composed and published in the volumes of the Academy or in the *Comptes Rendus* more than 500 memoirs, besides a multitude of reports on memoirs presented by others. In this immense mass of works, rapidly produced, many have a great value of their own; others present the initiatives of ideas and of methods, which have become already, or which will become at a later time, fertile. All bear upon the highest departments of Mathematics . . . .

. . . .

Unfortunately this haste in production did not leave him patience to bring his works to maturity. Each new way that presented itself to his mind occupied him exclusively; and in order to follow it he quitted that field in which he had begun to explore, even without taking time to see what it would lead to.

"For the sake of proceeding more rapidly, he almost always condensed his new researches in an unusual notation, which rendered them unintelligible to everybody but himself; and often he did not discover that these innovations disguised under some strange form results already known."

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* *l. c.*, pp. 151 – 152.

Klein criticises* this defect of Cauchy in severer language. "From 1835 onwards, Cauchy utilised for uninterrupted communications *Comptes Rendus* which had been appearing every week since the first July. His claim on that journal was of such a kind that the restriction of not having an article of more than four pages was introduced on his account. Notwithstanding this, his need to communicate herein was not long exhausted; by the side of it appeared the collected volumes brought out by himself; *Exercises de Mathematique* in several series, lectures, etc. All the publications of Cauchy have been brought out as 'collected works.' Frankly, the edition serves very little for rendering easy the peneteration into this jungle of writings."

The defect pointed out above hardly affects any of the books or bigger memoirs, each covering hundreds of pages, which Cauchy gave to the world in the first 20 years of his research activity. They are models of clarity and perfection of form. Even some of the smaller memoirs, e. g., that on polyhedra, have become classics. It was during his voluntary exile from France, in Switzerland, Italy or Bohemia, that the defect became exaggerated, and, on his return to France, could not be suppressed, specially because of his unemployment for many years and consequent anxiety to show his research activity. As Biot* points out, Cauchy's life was not as serene as that of Euler or Lagrange. Therefore we can only say that his achievements would have been greater and more dazzling, had his life been free from disturbing incidents.

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* See Klein's *Entwicklung*, l. c., pp. 72–73.

Cauchy had also his share of controversies regarding priority in discoveries. For example, he believed that he was the first to publish a proof of the reciprocity theorem which he claimed to be more rigorous than that found by Legendre and shorter than any of the proofs given by Gauss. This proof appeared in the *Bulletin des Sciences de M. Ferussac* in September, 1829. One can, therefore, imagine Cauchy's chagrin when in the third edition of his *Théorie des nombres*, which came out in 1830, Legendre "presents this demonstration as being the simplest of all and attributes it to M. Jacobi, without indicating any work in which that mathematician published that proof and which came out on a date before September, 1827".† Cauchy has stated his case. in a postscript at the end of the "Mémoire sur la thèorie des nombres" as follows: The matter "reduces to this very simple observation that the demonstration attributed by M, Legendre to M. Jacobi does not appear to have been published by one or the other of these two mathematicions before 1830. I am loth to wish to conclude that this demonstration had not been discovered by M. Jacobi at an earlier date. In the memoir of 1827, entitled 'De residuis cubicis commentatio numerosa', M. Jacobi, before enunciating theorems relating to the solution of indeterminate equations,...expressly says: 'In fontem uberrimum indici, e quo inter alia et demanare sequentia theoremata vidi'. The fertile source of which M. Jacobi speaks in this sentence, as declared by myself since (see in the *Bulletin des Sciences de M. de Ferussac*, the Memoir of September, 1829), is the consideration of the properties of the roots θ_h, θ_k,... of the auxiliary equation which proceeds from the solution of a binomal equation. Some of these properties had already led Gauss to important results contained in the last pages of his *Disquisitiones arithmeticae* and to his theorem on the solution of the equation $p = x^2 + y^2$. Thus the researches of M. Jacobi on the quadratic forms of prime numbers, and, one ought to say, those of mine may be considered as offering new developments of the beautiful theory expounded by M. Gauss. I shall add that the properties of the functions θ_h being supposed as known, it becomes very easy to obtain the demonstration referred to above. It is therefore natural to think that, on a date between 1827 and 1829, M. Jaccbi had found this proof and communicated it orally or in writing to M. Legendre. But what is the precise date of that communication ? This is a point on which I have no information and shall abide by the testimony of the illustrious mathematician of Koenigsberg".

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* *l. c.*, p. 154

† See the footnote on p 179 of the "Mémoire sur la théorie des nombres" (*Oeuvres*, s. 1, t. 3)

Cauchy's death was rather sudden; the following description* of his last days will give an idea of the suddenness:—"In the sitting of the Institute on the 4th of May, 1857, M. Cauchy read a second memoir on the employment in Astronomy of coefficient regulators, an employment which constitutes an artifice in Analysis on which he founded the greatest hopes and which he classed among the happiest of his discoveries. He was present at the sitting of the 11th of May, but was suffering from a bad cold; his family and friends perceived with grief that he appeared much weakened and that his features were changed. On Tuesday, the 12th of May, he repaired to his pleasant residence at Sceaux. He was unable to leave his room, yet nothing indicated his approaching end. He was continually occupied with the new developments in series, for which he was indebted to his regulator, and he completed the programme of his lectures at the Faculty of Sciences. On Thursday, the 21st of May, he conversed for some time with the Archbishop of Paris. His weakness increased on Friday, but he slept well that night; he awoke at three o'clock on Saturday morning, the 23rd of May, in a state of great feebleness, and in about half an hour expired, apparently without pain."

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* Proc. R. S. L., 1858

Long before his death, Cauchy's eminence had been recognized far and near by various institutions. In 1829, he was described as a Knight of the Legion of Honour, in 1832 he became a foreign member of the Royal Society of London, the exiled King Charles X made him a Baron in 1837 and before that time he had been admitted to the membership of the Academy of Sciences of Berlin, the Royal Society of Sciences of Prague, the Academy of St. Petersburgh, etc.

We have so far considered the work and conduct of Cauchy as a savant and as a teacher. His life as a private individual was exemplary. He was a good father of the family, a devoted brother and an affectionate son; at the age of 53 he studied Hebrew* in order to assist his aged father in some scriptural researches in which he was engaged. Cauchy was looked upon by all his contemporaries as a devout Catholic, as a man with a keen appreciation of his duty and as one who was inflexible in the discharge of that duty. Above all these things, his life outside the study or the Professor's chair, was one devoted to deeds of charity. The Mayor of Cauchy's commune stated in a letter published by Biot* that Cauchy would call on him several times a day to place before him the case of this poor child or that invalid soldier. When the Mayor suggested that Cauchy was spending far too much on charity, he received the reply that it was the Emperor who was paying, implying that all the income of Cauchy from theUniversity was spent on charity. Even, when dying, he discussed with the curate of his parish certain charitable schemes and on that ecclesiatic's asking him to take rest, Cauchy said that men pass away but their deeds remain. These were his last words.

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* Cauchy's aptitude for languages was remarkable. He could quote long passages from the Fathers in original. His command of English, German and Italian was very good. As regards French, he was not only an accomplished writer of prose in that language but also a good versifier. The following is the English translation (by my friend, Father Albert Schelvis, S. J.) of the verses recited by Cauchy at a joint meeting of the five academies where he spoke as the delegate of the Academy of Sciences :-

Your mind, my dear Alcippe, is verily distressed :
For of an evil spirit I seem to you possessed.
Indoors, you say, I morbid keep, a habit you deplore,
As with x's and with y's paper-sheets all day I score.
You would so like to see me quite otherwise employed,
With formulae and moduli ruthlesly destroyed,
My compasses and Algebra you 'ld throw against the wall,
To integrate and calculate seems silly after all. (For the original French, see M D' Ocagne's *Hommes et Choses de Science*, Paris, 1930, p. 121.)

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* l. c., pp. 158-159.

ABEL

〈1802—1829〉

CHAPTER III.

ABEL

The life-story of Niels-Henrik Abel, who was born in the small island of Finö near the south-west coast of Norway on the 5th August, 1802, and died on the 6th April, 1829, is a tragedy, relieved only by his epoch-making mathematical discoveries which, however, remained without any suitable recognition until the last year of his life. Born in a poor country of poor parents, no prince came forward to encourage him at any stage of his brief career, and his life was full of misery caused by poverty, by family worries and by unmerited neglect at the hands of those in power in the political world or the world of science. To understand this tragic phenomenon, it is necessary to know the political and economic condition of Norway in the early years of the 19th contry and also the condition of the family to which Abel belonged.

At the end of the 18th century, Norway was a possession of the Danish Crown and had been so for three centuries. When war broke out between England and France, Denmark and consequently, also Norway, became involved in it. From 1798 to 1802 there was the first war between England and Norway and again there were hostilities in 1807. The result of an effective blockade by the British of Norway along all its coasts for years, was disastrous for a country which could not live on its own resources: there was a great famine in 1811. Norway had to choose General Bernadotte as her king and peace came definitely in 1817 with Norway and Sweden united under the rule of Bernadotte.

The Abel family appears to have originated from the parish of Abild, District Tönder, in Schleswig. The first Norwegian ancestor, Mathias Abel, died in 1664 as an employee of the Prefecture of Throndhjem and from him descended a number of virile branches which produced many public functionaries, specially in the ecclesiastic line, all cultured and estimable. Abel's grand-father, Hans Mathias Abel, was born in 1738 and died in 1803 as the presbyter of the parish of Gjerrestad; his strength of character and devotion to duty were remarkable. In 1772, Sören Georg Abel, the father of Abel, was born; he was a talented, active man with some ambition, but he did not possess that strength of character which his father possessed. After being educated first at Helsingör (Elsinore) and then at Copenhagen, he passed his licenciate examination in Theology in 1792; two years later, he became assistant to his father at Gjerrestad and in 1799 was appointed to be Presbyter to the small parish of Finö near Stavanger. Here he entered on his duties with great zeal, founded popular libraries and took a leading part in an organization for the coastal defence for which work he was awarded the order of Danneborg. The mother of Abel, Anne Marie Simonsen, born in 1780 as the daughter of Niels Henrik Saxild Simonsen, a merchant of Risör, was a great beauty but lacking in firmness of character. It was of such parents that Abel was born as the second son, and at the time of his father's death in 1820 there were in all seven children to be provided for.

On the death of Abel's grandfather in 1803, Abel's father was appointed to the important living of Gjerrestad on the Gulf of Christiania and here Abel spent some happy years. Abel's earliest education was received from his father. In 1815, Abel entered the Cathedral School of Christiania, where brutality of manners still prevailed among the students and corporal punishment had not been abolished. During the first 2 or 3 years, Abel appeared as a mediocre student who was not much attracted towards his studies. But, about the end of 1817, an event took place which had remarkable consequences for Abel: the teacher of Mathematics was removed for having inflicted such a punishment on a student as caused his death, and Bernt Michael Holmboë (1795—1850) was appointed teacher in his place. Up to this time, the mathematical powers of Abel had remained latent. The teaching in Mathematics had been only 3 lectures a week out of a total of 40 lectures. With the year 1818, began a new period* for Abel. Holmboë was a man of solid knowledge without being a specialist. He was formerly a student at the Cathedral School and had then been marked by his humane character and idealism. He was an ardent patriot and was a member of the volunteer corps of students who fought against the Swedes when they invaded Norway in 1814 under the orders of Bernadotte. By Holmboë's influence, Abel was roused into intellectual activity. Holmboë's attention was particularly directed towards Abel to whom he used to give special lectures and in whom he believed to have discovered a potential genius. In the old registers of the school are found, in the midst of hackneyed remarks on Abel by the other teachers, the remarks of Holmboë which are full of enthusiastic praise. Thus in 1819, we have the remark: "remarkable mathematical genius"; in 1820: "the most remarkable genius who has a passion and insatiable ardour for Mathematics and will certainly become, if he lives, a great mathematician." Under the direction of Holmboë, Abel read by himself Euler's *Introductio*, *Institutiones Calculi Differentialis* and *Institutiones Calculi Integrali*. Thus armed, Abel read with ease, one after the other, the works of Lacroix Francoeur, Poisson, Gauss, Garnier and specially Lagrange. From a note-book of 192 pages, with the title "Exercises in Higher Mathematics by Niels-Henrik Abel", dating back to 1820, we see that he was at the time already well up in the theory of functions, and that he was specially interested in the theory of equations.

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* From the records of the school library, we learn that, whereas in 1817, Abel had sought out for study only such books from the library as were literary in character, viz. Danish accounts of voyages and historical works, there was a sudden change from 1818. In 1818 he took out Newton's *Arithmetica Universalis*; in 1819 Newton's *Opuscula Mathematica*, La Lande's *Astronomie*, Newton's *Philosophiae naturalis principia mathematica*; in 1820, Prony's *Mecanique philosophique*, Scherffer's *Institutiones mathematica*. And once he had commenced to take out books on mathematics, he did not borrow any book on literature.

In 1820, Abel's father died leaving to Abel the care of his mother and six children. The widow was in extreme poverty. But Abel locked at the future with great hopes and imagined that in due course, because of his special aptitude for Mathematics, he would get a University appointment which would suffice to support all his relatives. It was in the midst of such troubles and worries, only six months before his examination for admission to the University*, that Abel occupied himself with the solution of the general equation of the 5th degree. In July, 1821, Abel entered the University of Christiania which had been founded only about a decade back at great sacrifice by the Norwegian people and with great hopes of the benefits from it. Some months before his examination, his paper† on the solution of the general quintic had been sent by Hansteen to Copenhagen to Professor

* Examen Artium.

† The paper has been lost.

Carl Ferdinand Degen for being presented to the Royal Society of Sciences of Denmark.

Carl Ferdinand Degen was born at Brunswick in 1766 and had lived with his father at Copenhagen from the age of 5 years. He was a kind of encyclopaedist and not a specialist but had kept Mathematics in the forefront during his career as a student at the University of Copenhagen. After leaving the University he had served in secondary schools for many years and had become Professor of Mathematics at the University in 1814. Such a man was not aware of the previous researches of Ruffini and could not understand the true nature of the question which Abel had undertaken to solve; and it is almost certain that he was not in a position to judge the paper of Abel any better than Holmboë or Rasmussen (1768—1850), who was Professor of Mathematics and Physics at Christiania University, or Hansteen who was Professor of Astronomy and the highest mathematical authority at Christiania. The reply of Degen came on the 21st May, 1821, and, while expressing great admiration for Abel, Degen promised to communicate the memoir on receiving more details of the reasoning and a numerical example to illustrate it Thus Degen showed scepticism and caution without pointing out any defect. But, before the receipt of Degen's reply. Abel had himself discovered the defect which completely vitiated his solution.

At this stage it will be useful to consider briefly the previous history of the problem of the solution of the general quintic and for this purpose I shall use the words of Pierpont* in the introduction to his very interesting history: "After the brilliant successes of the Italian mathematicians of the 16th century, the contemporaries of Ferro, Tartaglia, Cardano and Ferrari could very well believe that the solution of the general equation of the 5th degree would follow soon. Still, more than a century elapsed before any one descried a method which would be applicable to that equation. First, in the year 1683 there appeared in the *Acta Eruditorum* a memoir on this subject by the naturalist and philosopher, Tschirnhaus. This memoir remained the single striking contribution to the subject in the 17th century, whilst in the 18th century the solution of the general equation of the 5th degree was straightways one of the famous mathematical problems of that century; Euler, Bezout, Lagrange, Vandermonde and Malfatti devoted to it searching investigations. Their works, although rich in valuable results, were of course in the main without success. The hope of a Leibintz and a Tschirnhaus gave place to a serious doubt. The resolvents of the 24th degree whose solution was required by the methods of Tschirnhaus, Euler and Bezout, were indeed replaced on account of the investigations of Lagrange, Vandermonde and Malfatti by one of the 6th degree. A further reduction, however, appeared out of sight, and consequently the solution of the general equation of the 5th degree appeared to be an impossibility.

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* "Zur Geschichte der Gleichung des V Grades (bis 1858)" (*Monatshefte für Math. u. Physik*, Bd. VI.)

It was an Italian of the name of Ruffini, physician and mathematician, who at the end of the century conceived the bold idea of proving this. His work, which appeared at Bologna in 1799 with the title *Teoria generale delle Equazioni* shall form for ever a landmark in the history of algebraic equations. That such a thorny undertaking would be fully successful in the first attempt was hardly to be expected and it was reserved first for the youthful Abel to furnish a completely rigorous proof and so prepare for the problem of the solution of this equation a correct formulation"

When Abel took admission into the University, "his state was precarious both as a student and as a resident; he was without money and was isolated at Christiania. Fortunately he had a friend whom he knew to be devotedly active. Holmboë obtained from the Academic Council a place for the young mathematician on the University Foundation (Regentsen) for poor students", and he also persuaded his colleagues to subscribe to a fund for Abel, thus relieving him to some extent of pecuniary troubles Abel passed in June 1822 the philosophical examination of the University and was thus freed from all attendance at lectures and was at liberty to devote himself to his mathematical studies alone. Higher and higher his reputation rose at his University. In two years he had carefully studied all the works in the various libraries at Christiania and had on many questions come to hold very personal views "All the scientific world of Christiania knew now his marvellous faculties : Holmboë and Professor Hansteen affirmed that he was already a remarkable mathematician"* and Professor Rasmussen wrote to his colleagues in Europe that Norway had given birth to a boy whose fame would dazzle the world one day.

In the beginning of 1823, Abel felt convinced that none of his teachers was capable of aiding him in his researches or judging their merit. He earnestly desired to leave the University and his country for some years in order to travel in Europe to visit the mathematicians there. He yearned to meet Laplace, Gauss, Poisson, Legendre, in fact all the great men whose works had made them familiar to him. This desire he hoped to satisfy by getting a subsidy from the University and he applied to the Academic Council for it and presented on the 23rd March, 1823, along with his application a big memoir, "a general exposition of the possibility of integrating all kinds of differentials." This memoir has been lost but we know that Abel thought that it deserved to be printed at the cost of the University. The Council referred the memoir to Rasmussen and Hansteen for report. But the report was not forthcoming before the beginning of the ensuing vacation and the Council itself did not want to have an early report. It is said that Rasmussen paid Abel sufficient money out of his own pocket to enable him, in default of his grand voyage, to pay a visit to Professor Degen at Copenhagen for which place Abel left in June with great joy and returned to Christiania in August.

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* See Ch. Lucas de Pesloüan's *N.—H. Abel, sa vie et son Oeuvre,* 1906, p. 10.

Abel awaited the report on his memoir but it was not forthcoming before December, 1823, when Hansteen and Rasmussen recommended a subsidy for travel abroad but did not recommend the publication of the memoir on account of reasons of economy. The recommendation for a subsidy was accepted on the condition that Abel should stay at the University for 2 years and learn foreign languages; the subsidy for this purpose was 200 spd., i. e., about 1000 francs a year. Abel persisted in his efforts to get his release to travel in Europe where, alone, he hoped to get recognition of his worth. He prepared for this purpose his paper "Mémoire sur les equations algebriques, ou l' on demontre l' impossibilite de la resolution de l' equation generale du cinquieme degre" which he got printed* at his own cost in spite of his poverty. He was so sure of the great value of this paper that he thought it to be a sure possport to the presence of the greatest mathematicians of Europe.

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* At the press of Gröndahl and son. The authors of the "Niels-Henrik Abel Memorial" point out the great improvement in the printing by the same press of Abel's *Oeuvres*, about 60 years later, this improvemet being a measure of the general progress of the people of Norway since Abel's time.

He sent a number of copies to Degen and through Schumacher of Hamburg he sent a copy to Gauss also. Schumacher refers to this paper in his letter to Gauss, dated the 23rd July, 1824. After much correspondence between the Academic Council and the Minister of Education, Abel was, by a royal decree, dated the 27th Aug., 1825, granted a subsidy of 600 spd. a year for travel abroad.

Before describing at some length the memoir about the impossibility, which we will do later, an idea may be given here of the mathematical activities of Abel during the two years of his stay at Chistiania before he started on his grand tour in 1825.

The memoir on the impossibility was by no means Abel's first publication. During the period in question, Abel published the following :—

* (1) "Méthode génèrale pour trouver des fonctions d' une seule quantité variable, lorsqu' une propriété des ces fonctions est exprimée par une équation entre deux variables." (*Magazin for Naturvidenskaberne*, Bd. 1, Christiania, 1823).

* (2) "Solution de quelques problémes á l' aide d' intégrales definies" (*Ibid*, Bd. 2, 1823).

* (3) "L' intégrale finie $\Sigma^n \phi x$ exprimeé par une intégrale definie simple" (*Ibid*, Bd. 2, 1825).

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* These appeared in Norwegian and their French translations are published in the *Oeuvres.*

(4) A memoir which appeared in the magazine quoted above in 1824 and which was later on withdrawn by Abel on account of a serious mistake.

In addition to the above published papers and the memoir which Abel sent to the Academic Council in 1824, Abel wrote a paper in June, 1824, at Hansteen's suggestion, on the calculation of the action of the moon on the pendulum. The paper was sent with Hansteen's approval to Schumacher for publication in the *Astronomische Nachrichten*; but it was not published as Schumacher detected a serious mistake in it which neither Holmboë nor Hansteen had been able to detect. This defect was a serions check to Abel who wanted to stand high in the estimation of such an influential and worthy man as Schumacher who was known throughout the whole of Europe.

A number of small papers which appear in the second volume of Abel's *Oeuvres*, edited by Sylow and Lie, are very probably of this period. Some of them are given below with their titles, along with a big memoir on the elliptic transcendentals which also belongs to this period :

(1) Les fonctions trancendantes $\Sigma\frac{1}{a^2}$, $\Sigma\frac{1}{a^3}$, $\Sigma\frac{1}{a^4}$ ... $\Sigma\frac{1}{a^n}$ exprimées par des intégrales définies." (pp. 1—6).

(2) "Sur l' integrale definie

$$\int_0^1 x^{a-1}(1-x)^{c-1}\left(l\,\frac{1}{x}\right)^{a-1}dx\text{''}$$

(pp. 7—13).

(3) "Sommation de la série $y=\phi(0)+\phi(1)x+\phi(2)x^2+\phi(3)x^3+\ldots+\phi(n)x^n$, n etant un nombre entier positif fini ou infini, et $\phi(n)$ une fonction algébrique rationnelle de n" (pp. 14—18).

(4) "Sur l' équation différentielle $dy+(p+q\,y+ry^2)\times dx=0$, oú p, q et r sont des fonctions de x seul" (pp. 19—25).

(5) "Sur l' équation différentielle $(y+s)\,dy+(p+q\,y+ry^2)dx=0$ (pp. 26—35).

(6) "Détermination d'une fonction au moyen d'une équation qui ne contient qu' une seule variable" (pp 36—39).

(7) "Propriétés remarquable de la fonction $y=\phi(x)$ determinée par l' équation $fy\,dy-dx\,\surd\{(a-y)\,(a_1-y)\times(a_2-y)\ldots(a_m-y)\}=0$ fy etant une fonction quelconque de y qui ne devient pas nulle ou infinie lorsque $y=a, a_1,\ldots a_m$" (pp. 40—42).

(8) Sur une propriété remarquable d' une classe trés étendue des fonctions transcendantes" (pp. 43—46).

(9) "Extension de la theorie precedente" (pp.47—54)

(10) "Sur la comparison des fonctions transcendantes" (pp. (55—66)

(11) "Sur les fonctions génératrices et leurs déterminants" (pp. 67—81).

(12) Sur quelques intégrales définies" (pp. 82—86)

(13) "Théorie des transcendantes elliptiques" (pp. 87—188).

Abel started about the end of September, 1825, on his grand tour of Europe and that tour and the benefits derived from it may be briefly described before we enter into details of visits to special men and special centres. Abel passed through Copenhagen where he made the acquaintance of V. Schmitten, the successor of Degen, who advised him to go to Berlin to get acquainted with Crelle. His next visit was to Altona via Lubeck and Hamburg, and he met at Altona Schumacher who informed him of the remark made by Gauss, viz. "Here is again one of those horrors", on seeing Abel's memoir on the impossibility of the solution of the general quintic. This news made Abel give up his previous intention of going to Göttingen and he went to Berlin instead. At Berlin he stayed for six months and secured the friendship and patronage of Crelle who was then thinking of bringing out a mathematical journal which came out as *Journal für die reine und angewandte Mathematik*. In all Abel contributed over a dozen important papers to *Crelle's Journal* in the years 1826-1829. From Berlin, Abel went to Freiberg where he worked for sometime in great peace and wrote his memoir "Sur l' intégration de la formule différentielle $\frac{\rho dx}{\surd(R)}$, R et ρ étant des fonctions entieres" (*Oeuvres*, t. 1, pp. 104—144.) He left Freiberg and in 3½ months made a tour of Vienna, Prague, Trieste, Venice, Padoa, Verona, Botzen and Zurich. He went from Zurich to Paris where he staid for about six months practically in isolation from the great mathematicians of that city with none of whom could he establish real personal touch. Here he finished his great memoir, "Mémoire sur une propriété generale d' une classe trés étendue de fonctions transcendantes" in October, 1826, and sent it to the Institute soon after; the memoir was destined to remain neglected for years and then to be crowned with the Grand Prize in 1830. He left Paris in December, 1826, and arrived at Berlin after passing through Brussels, Aix-la-Chapelle, Cologne and Magdeburg. Leaving Berlin in the beginning of May, 1827, he arrived at Christiania via Copenhagen after 20 months of absence. He arrived "poor in money, rich in knowledge but without having obtained in Europe the least official recognition of his genius."* The one great friend he made was Crelle who tried his best to secure him with a post at Berlin.

The first meeting between Abel and August Leopold Crelle (1780—1856), a man of high social position, Geheimer Oberbaurat, builder of several of the most important roads in Prussia and the first railroad in that country, a self-taught mathematician who was penetrated through and through with the sincerest conviction of the importance of Mathematics for life, has been variously reported. Abel himself reports in a letter to Professor Hansteen as follows: "It took a long time for me to make clear to him what my real aim was in visiting him, and it appeared first as if the visit would have a lamentable end; however, 'I mustered courage when he asked me what I had already read in Mathematics. When I had mentioned to him the writings of some of the most eminent mathematicians, he became very attentive and, as it appeared to me, really pleased. He entered into a long talk with me on many difficult matters which had not till then been settled, and as, whilst our talk turned to the higher equations, I said to him that I had proved the impossibility of the general equation of the 5th degree, he would not believe it and said that he would oppose it. I presented to him therefore a copy (of the memoir); however, he said he could not understand the reason of many of my conclusions. The same has been said by many others and for that reason I have undertaken a remodelling." Mittag-Leffler writes that as reported by Crelle himself to Weierstrass this is what happened. "Crelle was at the time of Abel's visit an Examiner at the Polytechnic Institute at Berlin, a post in which he found no particular pleasure. Then one fine day entered into his room a blonde young man with very embarassed, very young and very intelligent appearance. Crelle believed that the man wanted to submit to an examination for admission to the Polytechnic Institute, and explained that for that purpose a number of separate examinations would be necessary. Then the young man finally opened his mouth and explained: 'Not examination, only Mathematics" Crelle noted that he had to deal with a foreigner and attempted to speak to him in French which, as it turned out, Abel could speak well although with some difficulty. On Crelle's questioning him about his studies, Abel said he had read Crelle's own recently published work on 'Analytical Faculties' which, regardless of many mistakes, had appeared to him most interesting. With the mention of many mistakes, Crelle became most attentive, and now began a talk which was to lead to such close relation between Abel and Crelle. As before, Holmboë and still more Hansteen, so also Crelle was quite incapable of understanding Abel's work. Crelle himself has furnished a drastic proof of this. Abel's work on the binomial series is found published in the first volume of Crelle's own journal and was personally translated by Crelle from the French Manuscript into German. This did not hinder Crelle from publishing after Abel's death in the 4th and 5th volumes of his journal a memoir on the subject of binomial series in which he held the pre-Abel point of view and showed himself fully ignorant of the fact that the whole question had received its final settlement at the hands of Abel. Even if, like the Norwegian friends of Abel, Crelle understood very little of his work, he understood Abel's genius and with that understanding became Abel's active friend and protector. Already at the first visit of Abel, Crelle had mentioned his plan of bringing out a great German journal. The acquaintance with Abel and the prospect of securing him as a collaborator, hastened the realization of the plan."* According to Mittag-Leffler, the fact that *Crelle's Journal* became a world-organ is essentially due to Abel whose works from the beginning conferred on the journal the highest possible rank.

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* Pesloüan, l. c., p 88.

On learning from Schumacher the slighting words uttered by Gauss with regard to the memoir on the imposibility of the general solution of the quintic, Abel's sentiments towards Gauss underwent a change and he no longer cared to go to the "intolerable Gauss" against whom Abel developed "a certain animosity which was not without jealousy" † He freely indulged in diatribes against Gauss of whom he writes from Berlin : "It is extraordinary how the young mathematicians carry Gauss to the pinnacle. He is for them the substance of all mathematical perfection, but, although he is really a great genius, it is also certain that he writes badly. Crelle says that all that Gauss writes is a horror, for it is so obscure that it is impossible for any one to understand it".‡ So the original project of going to Göttingen and then to Paris was modified and Abel went to Paris and arrived there in the second half of July, 1826. He found board and lodging with a family at a cost of 120 francs a month, although the room was very bad and he had only two meals a day. In the beginning he was full of pleasure at finding himself in Paris, the focus of eminent mathematicians whom he hoped to meet and whose recognition of his worth he hoped to secure. However, when he went to the house of Legendre he could have only a few words with the great man as he was just entering into his carriage to go out. The desire to have a great work ready with which he could approach Cauchy made Abel wait until the end of October when his work was finished. But his efforts to make Cauchy take interest in that work proved to be unsuccessful and it was a comparatively unknown mathematician, Hachette, through whose efforts Fourier communicated the memoir to the Institute. The memoir was referred to Legendre and Cauchy; with the latter it remained for several years and was supposed to have been lost by him. However, the energetic representations of the Norwegian consul at Paris after Abel's death resulted in a search being made for the memoir which was found in 1830. In the same year on the 24th July, Abel was awarded jointly with Jacobi the grand prize of the Institute, although the memoir itself was published by the Institute later in 1841. So, recognition did not come to Abel at Paris in his lifetime. He found the Paris mathematicians cold towards him, self-centred, more desirous of instructing than of learning. When Abel left Paris, he left without having made a single friend.

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* *Crelle's Journal*, Bd. 157.

† See Pesloüan's book, p. 47

‡ See the letter of Abel to Hansteen sent from Berlin on 5. 12. 1825. Also published on p. 11 of "Correspondance d' Abel", in "Niels-Henrik Abel Memoiral".

Klein's remarks on the grand tour of Abel are of great interest: the following* may be quoted here:—

"This tour has been for Abel of decisive importance; here originated his chief ideas, or better said, Abel was compelled by contact with the mathematical neighbourhood which newly presented itself to him to give the ideas form in order to follow them out; just as an oversaturated solution is suddenly brought to crystallization by the slightest external shock."

Again Klein says about the relations between Abel and Crelle: "Of the greatest importance it was that soon after his arrival (at Berlin) he met Crelle . . . . . . In his house the man so badly treated by Fate found the friendly reception and the encouraging talk which his timid nature needed. Abel on his own side responded to the offered kindness with great confidence. With true fiery zeal he entered into Crelle's proposal so that the first volume of the journal could bring forward 6 papers and remarks from his pen; all of them were written down during the short period of his first stay at Berlin or indeed nearly completed. Abel's whole nature bloomed; in his otherwise sorrowful life these few months denote a period of pure happiness."

We proceed now to consider in detail first the memoir on the impossibility of the solution of the general quintic by radicals alone and then the other important papers published in *Crelle's Journal* before Abel's return from his grand tour:—

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* See Klein's *Entwicklung*, l. c., p. 101.

(I) and (II)

(I) "Mémoire sur les équations algébriques, où l' on démontre l' impossibilite de la resolution de l' équation générale du cinquieme degré" (Gröndahl, Christiania, 1824; *Oeuvres*, t. 1, pp. 28—34).

(II) "Démonstration de l' impossibilite de la resolution algébrique des équations générales qui passent le quatrieme degré" (*Crelle's Journal*, Bd. I, 1826, pp. 65—84; *Oeuvres*, t. 1, pp. 66—87).

We begin with (II), a summary of which was prepared and published by Abel himself* in the *Bulletin des sciences de Ferussac*, 1826, and quote from that summary† the following outline of the demonstration:—

"The author proves, in this memoir, that it is impossible to solve algebraically the general equation of the 5th degree; for, every algebraic function of the coefficients of the proposed equation when substituted for the unknown leads to an absurdity.

"In the first paragraph the author investigates the general expression of the algebraic function of several quantities, after the definition that an algebraic function results from (1) additions (2) multiplications, (3) divisions and (4) extractions of roots of which the exponents are *prime* numbers. The subtractions, the raising to powers and the extractions of roots with *composite* exponents are included in the preceding operations. Whence it follows (1) that every rational and integral function of the quantities x_1, x_2, x_3, etc., that is to say, every function which can be formed by means of the first two operations mentioned above can be expressed by a sum of a *finite* number of terms of the form

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* See his letter of 24.10.1826 to Holmboë.

† *Oeuvres*, t. 1, pp. 88—94.

$$A\ x_1^{m1}\,x_2^{m2}\ldots,$$

A being a constant and m_1, m_2,... being integral numbers; (2) that every rational function of the same quantities, that is to say, every function that can be formed by means of the first three operations can be expressed by a quotient of two integral functions; (3) that every algebraic function can be formed by repetitions of the operations indicated by

$$p' = f\left(x_1,\ x_2, \cdots\ p_1^{\frac{1}{n_1}},\ p_2^{\frac{1}{n_2}}, \ldots\right) \tag{1}$$

where f denotes a rational function of the quantities inside the brackets; p_1, p_2,... denote rational functions of the quantities x_1, x_2,... and n_1, n_2,... denote rational functions of prime numbers. One shall call, for the sake of brevity, *algebraic function of the first order* a function such as p'. If now one forms a new function in which functions of the first order enter in the same manner as p_1, p_2,... enter in p', one shall have an *algebraic function of the second order*; and in general a function of the μth order shall be that which contains functions of all orders up to the $(\mu-1)$th order combined together *algebraically*. ... Besides if this function of the μth order contains m quantities of this order, one says that it is of the mth degree; and by designating it by v one can put

$$v=q_0+p^{\frac{1}{n}}+q_2\,p^{\frac{2}{n}}+\cdots+q_{n-1}\,p^{\frac{n-1}{n}},\quad(2)$$

that is to say, one has this first theorem: *Every algebraic function v of order μ and degree m can be represented by the formula* (2), *where n is a prime number, q_0, q_2, q_{n-1} are algebraic functions of order μ and of degree $m-1$ at the utmost and p is an algebraic function of order $\mu-1$ such that it is impossible to express $p^{\frac{1}{n}}$ by a rational function of p, q_0, $q_2 \cdots q_{n-1}$.*" In the second paragraph, the following theorem is proved: *If an algebraic equation is soluble algebraically, one can always give to the root such a form that all the algebraic expressions of which it is composed can be expressed by the rational functions of the roots of the proposed equation.* In order to prove this theorem, Abel first shows that if

$$\phi(x_1, x_2, \ldots, y)=0$$

be an equation with y as the unknown and the coefficients are rational functions of x_1, $x_2, \ldots$, the equation, if soluble algebraically, will have its roots

$$y_1=q_0+p^{\frac{1}{n}}+q_2\,p^{\frac{2}{n}}+\ldots+q_{n-1}\,p^{\frac{n-1}{n}},$$

$$y_2=q_0+a\,p^{\frac{1}{n}}+a^2\,q_2\,p^{\frac{2}{n}}+\ldots+a^{n-1}\,q_{n-1}\,p^{\frac{n-1}{n}},$$

. . . .

$$y_n=q_0+a^{n-1}\,p^{\frac{1}{n}}+a^{n-2}q_2\,p^{\frac{2}{n}}+\ldots+a q_{n-1}p^{\frac{n-1}{n}},$$

where a is an imaginary root of unity.

In the third paragraph use is made of the following results of Cauchy (*Journal de l' Ecole Polyt.*, Cahier XVII):

(1) the number of the values of a rational function of n quantities can not diminish below the greatest prime number contained in n without becoming equal to 2 or 1; (2) every rational function which has two different values shall have the form

$$p+q\,(x_1-x_2)\,(x_1-x_3)\,..(x_2-x_3)..\,(x_3-x_4)\ldots\ldots$$

and if it contains 5 quantities it shall become

$$p+q\,(x_1-x_2)\,(x_1-x_3)\,(x_1-x_4)\,(x_1-x_5)$$
$$\times(x_2-x_3)\,(x_2-x_4)\,(x_2-x_5)\,(x_3-x_4)\,(x_3-x_5)\,(x_4-x_5)$$

where p and q are invariable functions.

"One proves that every rational function of 5 quantities which has five different values can be put in the form

$$v=r_0+r_1x+r_2\,x^2+\ldots+r_4\,x^4$$

where $r_0, r_1,\ldots r_4$ are invariable functions, and x is one of the five quantities in question. By combining this equation with the equation

$$(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)$$
$$=x^5-ax^4+bx^3-cx^2+dx-e=0$$

one can deduce the values of x in the form

$$x=s_0+s_1 v+s_2 v^2+s_3 v^3+s_4 v^4,$$

s_0, s_1,...being invariable functions of x_1, x_2,...

"Finally one arrives at this theorem:—

If a rational function of several quantities $x_1,x_2,...$ *has m different values, one can always find an equation of degree m of which all the coefficients are invariable functions of* x_1, x_2,...*and which has the m values of the function for roots; but it is impossible to find an equation of the same form of a lower degree which shall have one or more of these values for roots.*"

In the fourth and last paragraph it is proved by means of the theorems established in the preceding paragraphs that it is impossibe to solve algebraically the general equation of the 5th degree.

"In fact by supposing that the general equation of the 5th degree is soluble algebraically, one can by virtue of the theorem (1) express all the algebraic functions, of which a root is composed, by rational functions of the roots; therefore because it is impossible to express a root of a general equation by a rational function of the coefficients it is necessary that one should have

$$R^{\frac{1}{m}} = v$$

where $R^{\frac{1}{m}}$ is one of the functions of the first order which are found in the expression of the root, R being a rational function of the coefficients of the proposed equation, that is to say, an invariable function of the roots, and v a rational function of the same roots. This equation gives

$$v^m - R = o\,;$$

and for v, m different values, resulting from the change of the roots among themselves. Now the number of the values of a rational function of 5 variables ought to be a divisor of the product 2. 3. 4. 5 ; it is therefore necessary that m, which is a prime number, should be one of the three numbers 2, 3, 5 ; but according to the theorem of Cauchy quoted above the number 3 should be excluded, and by consequence there remain for m the two values 5 and 2.

"I. At the outset let $m=5$; one shall have, after what one has seen previously,

$$v = R^{\frac{1}{5}} = r_0 + r_1 x + r_2 x^2 + r_3 x^3 + r_4 x^4\,;$$

and hence

$$x = s_0 + s_1 R^{\frac{1}{5}} + s_2 R^{\frac{2}{5}} + s_3 R^{\frac{3}{5}} + s_4 R^{\frac{4}{5}}\,,$$

s_0, s_1, ... being like R invariable functions of the roots. This value gives, according to what has been established in the second paragraph, for $s_1 R^{\frac{1}{5}}$ a rational function of the roots, namely,

$$s_1 R^{\frac{1}{5}} = \frac{1}{5}(x_1 + a^4 x_2 + a^3 x_3 + a^2 x_4 + a x_5) = z,$$

a being an imaginary root of the equation $a^5 - 1 = 0$; but this is impossible as the second member has 120 different values, whereas it ought to be the root of the equation

$$z^5 - R s_1^5 = 0$$

which is only of the 5th degree. Therefore m cannot be equal to 5". Abel shows that $m = 2$ also leads to an absurdity. Thus the impossibility is proved.

The proof given by Abel in (II) did not satisfy many mathematicians; and in fact the Prague Royal Society of Sciences declared in 1832 both Abel's proof and Ruffini's proof to be unsatisfactory and announced a prize for a fully satisfactory proof. Even Abel notes the imperfection of his proof when he says: "The first, and, if I do not deceive myself, the only one who before me sought to prove the impossibility of the algebraic solution of the general equation, was the mathematician Ruffini; but his memoir is so complicated that it is very difficult to judge of the justice of his reasoning. It appears to me that his reasoning is not always satisfactory. I believe that the demonstration which I have given in the first volume of this journal leaves nothing to be desired on the side of rigour; but it has not all the simplicity of which it is susceptible. In seeking to solve a more general problem I have arrived at another demonstration, founded on the same principles, but simpler".* This simpler proof was not published by Abel and it was given by Kronecker in his paper† "Vereinfachung des Abelschen Beweises".

If we place by the side of (II) which covers in the *Oeuvres* 22 pages, the first proof, given in (I), a paper of hardly six pages, we can realize the difficulty which mathematicians felt in accepting Abel's opinion of that proof as sound and convincing. Its defects have been pointed by Pesloüan almost in the following words:—As a man of genius, Abel seems to have failed to realize the necessity of a syllogistic development of his reasoning which would be felt by men of mediocre intelligence. "In this work on the equation of the 5th degree, the absence of all useless development renders this impression particularly strong. The reasoning is neatly divided into two parts: the one appears to be an exposition of the problem, it consists in determining the form of the algebraic expression which can satisfy an equation of the 5th degree. The result at which Abel arrives is that all the irrational quantities which enter in the expression can be expressed as rational functions of the roots. This result leads naturally to the followign study: Once it is determined that the irrational quantities are at the utmost of the 5th degree and that they are equal to a rational function of the roots, it is necessary that by permuting all the roots, the irrational quantity should take all its values. This study checks without doubt the mathematician before he has known the theorem of Cauchy: 'The number of the values of a rational function of n quantities cannot be less than at the utmost the greatest prime number contained in n without being 2 or 1'. This theorem furnishes immediately the result. The rational function of 5 roots can have only 5 or 2 values. He proves that none of these two will suit; therefore the algebraic solution of the equation is impossible. Such is the brief analysis of this very brief paper"*.

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* See *Oeuvres*, t. 2, p. 218.

† *Monatsberichte der Akademie der Wissenschaften zu Berlin*, 1879, pp. 205 onwards; also *Werke*, Bd. 4, pp. 75-80.

(III)

"Résolution d' un probléme de Mecanique"

(*Crelle's J.*, Bd. I., 1826; *Oeuvres*, t. 1, pp. 97—101)

This paper is rightly believed to be one of the earliest publications to contain what is now called an *integral equation*. Abel introduces the subject thus: "Let B D M A be a curve whatever. Let B C be a horizontal straight line and C A a vertical straight line. Let us suppose that a particle urged by gravity moves on the curve, a point D whatever being its point of departure. Let τ be the time which has passed when the particle is at the given point A, and let a be the height E A.* The quantity τ shall be a certain function of a which shall depend on the form of the curve. Reciprocally, the form of the curve shall depend on this function. We proceed to examine how with the help of a definite integral, one can find the equation of the curve for which τ is a given continuous function of a."

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* Ch. Lucas de Pesloüan : *N.—H. Abel, Sa Vie et Son Oeuvre*, pp. 21—22.

In fact, if AM$=s$, AP$=x$ and t is the time taken by the particle in running through the arc D M, then

$$\phi(a)=\int_0^a \frac{ds}{\sqrt{(a-x)}}$$

Abel solves this and gets

$$s=\frac{1}{\pi}\int_0^x \frac{\phi(a)\,da}{\sqrt{(x-a)}}.$$

Generally from

$$\phi(a)=\int_0^a (a-x)^{-n}ds,\ 0<n<1,$$

Abel gets

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* D E and M P are horizontal; E and P are on C A.

$$s=\frac{\sin n\pi}{\pi}\int_{0}^{x}\phi(a)\,(x-a)^{n-1}da.$$

(IV)

"Sur L' Intégration de la formule différentielle $\frac{\rho dx}{\surd(\text{R})}$, R and ρ étant des fonctions entiéres"(*Crelle's J.*, Bd. I, 1826; *Oeuvres*, t. 1, pp. 104—144).

This memoir is called the memoir of Freiberg where it was completed in 1826. Its object is "to find all the differentials of the form $\frac{\rho dx}{\surd(\text{R})}$,(where ρ and R are integral functions of x) of which the integrals can be expressed by a function of the form

$$\log\frac{p+q\surd(\text{R})}{p-q\surd(\text{R})}\text{ ,"}$$

where p and q are integral functions of x.

Abel's solution is the following: "When it is possible to find for ρ an integral function such that

$$\int\frac{\rho dx}{\surd(\text{R})}=\log\frac{y+\surd(\text{R})}{y-\surd(\text{R})}$$

the continued fraction resulting from $\surd(\text{R})$ is periodic and of the form

$$\surd(\text{R})=r+\cfrac{1}{2\mu+\cfrac{1}{2\mu_1+\ldots+\cfrac{1}{2\mu_1+\cfrac{1}{2\mu+\cfrac{1}{2r+\cfrac{1}{2\mu+\cfrac{1}{2\mu_1+\ldots}}}}}}}$$

and reciprocally when the continued fraction of $\surd(R)$ has this form it is always possible to find for ρ an integral function which satisfies the equation

$$\int \frac{\rho\,dx}{\surd(R)} = \log \frac{y+\surd(R)}{y-\surd(R)}.$$

The function y is given by

$$y = r + \frac{1}{2\mu+}\;\frac{1}{2\mu_1+}\;\frac{1}{2\mu_2+}\dots\frac{1}{2\mu+}\;\frac{1}{2r}$$

Some examples are given by way of illustration :

$$\int \frac{(3x+a)\,dx}{\surd(x^2+ax)^2+ex} = \log \frac{x^2+ax+\surd R}{x^2+ax-\surd R}$$

(V)

"Mèmoire sur une propriété généralé d'une classe trés-étendue des fonctions transcendantes" (Presented* to the Academy of Sciences of Paris on 30. 10. 1826; published in the Memoirs in 1841 ; *Oeuvres*, t. 1, pp., 145-218).

In the introduction to this memoir which contains the "theorem of Abel", the following lines are found : "A function of which the differential coefficient is rational has, as one knows, the property that one can express the sum of an arbitrary number of similar functions by an algebraic and logarithmic function whatever be moreover the variables of these functions. Similarly, any elliptic function, that is to say, a function, of which the differential coefficient contains no other irrationality than a radical of the second degree under which the variable does not exceed the fourth degree, shall have still the property that one can express the sum of any number of similar functions by an algebraic and logarithmic function provided that one establishes between the variables of these functions a certain algebraic relation. This analogy between the properties of these functions has led the author to seek if it shall not be possible to find analogous properties of the most general functions and he has arrived at the following theorem :—'If one has several functions of which the differential coefficients can be roots of *one and the same algebraic equation*, of which all the cœfficients are *rational* functions of one and the same variable, then one can always express the sum of an arbitrary number of similar functions by an *algebraic* and *logarithmic* function, provided that one establishes between the variables of the functions in question a certain number of algebraic relations'. The number of these relations does not depend on the number of the functions but only on the nature of the particular functions which one considers. Thus, for example, for an elliptic function this number is 1 ; for a function of which the differential cœfficient contains no irrationalities other than a radical of the 2nd degree under which the variable does not exceed the 5th or 6th degree, the number of the necessary relations is 2, and so on. The same theorem subsists still when one supposes the functions multiplied by rational numbers whatever positive or negative. One deduces further the following theorem: 'One can always express the sum of a *given* number of functions, which are multiplied each by a rational number, and of which the variables are arbitrary, by a similar sum of *determinate* number of functions of which the variables are algebraic functions of the variables of the given functions."

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* The actual writing of the memoir was begun at Paris on the 9th August 1826 and completed there in about 2 months.

At the end of the memoir is given the application of the theory to a particular class of functions, viz, those which are expressed as integrals of differential forms which contain no other irrationality than a radical whatsoover".

(VI)

" Recherches sur la série $1 + \frac{m}{1} x + \frac{m(m-1)}{1.2} x^2 + \frac{m(m-1)(m-2)}{1.2.3.} x^3 + \ldots$"

(*Crellé's J.* Bd. 1, 1826 ; *Oeuvres,* t. 1, pp. 219—250).

"The aim of this memoir* is to attempt to fill up a lacuna by the complete solution of the following problem : To find the sum of the series

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* Klein says of this memoir : "It is Abel's most important contribution to the exact foundation of Analysis .....This work is to be regarded as the pure fruit of his stay at Berlin where he came to know in Crelle's library Cauchy's Cours d' Analyse " The memoir was begun at Berlin and was continued at Freiberg and was finished in several months......It appears to be isolated in the chain of Abel's researches.

$$1 + \frac{m}{1}x^2 + \frac{m(m-1)}{1.\ 2}x^2 + \frac{m(m-1)\ (m-2)}{1.\ 2.\ 3.}x^3 + \ldots$$

for every real or imaginary value of x and m for which the series is convergent"(See p. 221 of the *Oeuvres*, t, 1). The results obtained by Abel are the following:— (1) When the series $1 + \frac{m+ni}{1}(a+bi) + \frac{(m+ni)\ (m-1+ni)}{1.\ 2.}(a+bi)^2 + \ldots$ is convergent it has for sum

$$\left\{(1+a)^2 + b^2\right\}^{\frac{m}{2}} e^{-n \tan^{-1} \frac{b}{1+a}}$$

$$\times\left[\cos\left\{m \tan^{-1} \frac{b}{1+a} + \frac{n}{2}\log\left(1+a^2+b^2+2a\right)\right\}\right.$$

$$\left.+\ i \sin\left\{m \tan^{-1} \frac{b}{1+a} + \frac{n}{2}\ \log\ (1+2a+a^2+b^2)\right\}\right].$$

(2) The series is convergent for every value of m and n when $(a^2+b^2)^{\frac{1}{2}}$ is less than 1. If $(a^2+b^2)^{\frac{1}{2}}$ is equal to 1 the series is convergent for every value of m comprised between -1 and ∞ if one has not at the same time $a = -1$. If $a = -1$, m ought to be positive. In all other cases the series is divergent.

Among other things the memoir contains the famous theorem of continuity: If a power series converges in a given point A of the circle of convergence then is the series uniformly convergent on the radius directed to that point so that the function represented by the power series inside the circle of convergence tends to a limit equal to the sum of the series at A when the point moves along the radius O A to A.

(VII)

"Sur quelques intégrales définies" (*Crelle's J.*, Bd. 2, 1827; *Oeuvres*, t. 1, pp. 251—262) In this are studied a number of definite integrals which satisfy linear differential equations of the 2nd order.

(VIII)

"Recherches sur les fonctions elliptiques" (*Crelle's J.*,* Bde. 2 and 3, 1827 and 1828; *Oeuvres*, t. 1, pp. 263-388)

Klein† says of this memoir: "This is the great fundamental publication with which for the mathematical public, since Gauss had surely kept back his results, the theory of elliptic functions—in contradistinction with the Legendre's theory of elliptic integrals—begins". Abel defines the inverse function $x=\phi(\alpha)$ by the equation

$$d\alpha=\frac{dx}{\sqrt{\{(1-c^2x^2)(1+e^2x^2)\}}}=\frac{dx}{s}.$$

and the two allied functions

$$f(\alpha)=\sqrt{\{1-c^2\phi^2(\alpha)\}},\quad F(\alpha)=\sqrt{\{1+e^2\phi^2(\alpha)\}}.$$

He defines the two quantities

$$\frac{w}{2}=\int_0^{\frac{1}{c}}\frac{dx}{s},\quad \frac{\omega i}{2}=\int_0^{\frac{i}{e}}\frac{dx}{s}.$$

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* 1st part appeared on 20-9-1827, the 2nd part on 26-5-1828.

† Klein's *Entwicklung*, &c., p. 105

After having determined the addition formula $\phi(\alpha+\beta) = \frac{\phi(\alpha) f(\beta) F(\beta) + \phi(\beta) f(\alpha) F(\alpha)}{1 + e^2 c^2 \phi^2(\alpha) \phi^2(\beta)}$, he shows the property of the double periodicity of $\phi(\alpha)$, determines the zeros and the poles and all the roots of the equation

$$\phi(\alpha) = \phi(\beta).$$

He passes to the mutiplication formula and proves that $\phi(n\beta)$, $f(n\beta)$, $F(n\beta)$ are rational functions of the quantities $\phi(\beta)$, $f(\beta)$, $F(\beta)$. He arrives immediately at the inverse problem, viz. that of division. He proves that the equation,

$$\phi(\alpha) = \frac{P\left(\frac{\alpha}{2n+1}\right)}{Q\left(\frac{\alpha}{2n+1}\right)}$$

is of degree $(2n+1)^2$ in $\phi\left(\frac{\alpha}{2n+1}\right)$. It is entirely soluble algebraically as soon as one knows the constants $\phi\left(\frac{w}{2n+1}\right)$ and $\phi\left(\frac{wi}{2n+1}\right)$. These constants depend on an equation of degree $(2n+2)\times n$ which resolves itself into an equation of degree $(2n+2)$ and equations of degree n. These equations of degree n are soluble algebraically; in general this is not the case with the equation of degree $(2n+2)$. In some cases we have algebraical solubility, e. g. when $e=c=1$, the case of the lemniscate. Before taking this particular case, Abel, with the aid of the equation

$$\phi(a)=\frac{P\left(\frac{a}{2n+1}\right)}{Q\left(\frac{a}{2n+1}\right)}$$

in calculating the sum and the product of roots, passes to the development in series and infinite products. He introduces at this moment the series in terms of sines and cosines as well as in exponentials.

The problem of the division of the arc of the lemniscate is then taken for the case of a prime divisor of the form $4n+1$; Abel insists on the result—identical with that of Gauss—of the possibility of making the division by square roots (*i. e.* geometrically with the compass) when the divisor is a prime of the form 2^n+1.

In the second part of the memoir, Abel is occupied with the transformation; he does not always put the problem in a general manner and the actual text presents itself as the verification of a given formula. It is therefore difficult to know how the author came to discover this formula. That which it is necessary to remark is this that the constants which it contains are the different roots of the equation of division for a certain divisor. Finally, the last paragraph of the memoir proper treats of the generalized Euler's transformation; viz.

$$\frac{dy}{\sqrt{\{(1-y^2)(1+\mu y^2)\}}}=a\frac{dx}{\sqrt{\{(1-x^2)(1+\mu x^2)\}}}$$

Abel finds that a ought to be rational or the product by i of the square root of a rational number*.

There is at the end of the memoir proper the "Addition au mémoire precedent" (pp. 385—388) which contains the following lines in the beginning:—"Having finished the preceding memoir on the elliptic functions, a note on the same functions by M. C. G. J. Jacobi, inserted in the No. 123 of 1827 of the journal which is edited by M. Schumacher and has the title "Astronomische Nachrichten", has been seen by me. M. Jacobi gives the following theorem: Let p be an odd number and θ' such an angle that one has, by designating the integral

$$\int \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}} \text{ taken from 0 to } \theta \text{ by } F(k,\theta),$$

$$F(k,\theta') = \frac{1}{p} F(k, 90^\circ),$$

and in general $\theta^{(m)}$ an angle such that one has

$$F(k, \theta^{(m)}) = \frac{m}{p} F(k, 90^\circ);$$

and let ψ be determined by the equation

$$\tan(45^\circ - \tfrac{1}{2}\psi)$$

$$= \frac{\tan\frac{1}{2}(\theta' - \theta).\tan\frac{1}{2}(\theta''' + \theta).\cdots\tan\frac{1}{2}(\theta^{(p-2)} \pm \theta)}{\tan\frac{1}{2}(\theta' + \theta)\tan\frac{1}{2}(\theta''' - \theta)\ldots\tan\frac{1}{2}(\theta^{(p-2)} \mp \theta)} \times$$

$$\tan(45^\circ \mp \tfrac{1}{2}\theta);$$

then one shall have

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* This account of Abel's memoir proper is reproduced from Pesloüan's book, pp. 76—79.

$$F(k, \theta) = \mu . F(\lambda, \psi).$$

"It is necessary to take the upper sign if p is of the form $4n+1$, and the lower sign if p is of the from $4n-1$. ψ ought to be taken between $\frac{m}{2}\pi$ and $\frac{m+1}{2}\pi$ if θ lies between $\theta^{(m)}$ and $\theta^{(m+1)}$. The constants μ and λ are determined in different manners. One has for example

$$\mu = \frac{1}{2\,(\operatorname{cosec}\theta' - \operatorname{cosec}\theta''' + \,.\, \mp \operatorname{cosec}\theta^{(p-2)} \pm \frac{1}{2})}$$

$$\lambda = 2k\,\mu(\sin\theta' - \sin\theta''' + \ldots \mp \sin\theta^{(p-2)} \pm \tfrac{1}{2})$$

"This elegant theorem which M. Jacobi gives without demonstration is contained as a particular case in the formula (227) of the preceding memoir, and at the base is the same as formula (270). We proceed to prove this".

(IX)

"Sur les fonctions qui satisfont á l' équation

$$\phi(x) + \phi(y) = \psi(xfy + yfx)"$$

(*Crelle's J.*, Bd. 2, 1827; *Oeuvres*, t. 1, pp. 389—398)

The equation is satisfied when $f(y) = \frac{1}{2}\,y$ and $\phi\,(x) = \psi\,(x) = \log x$; also when $f(y) = \sqrt{\{1-y^2\}}$ and $\phi(x) = \psi(x) = \sin^{-1}x$. Abel's object is to find in what other ways the equation can be satisfied. He proves that the most general functions satisfying the equation are the following:—

$$\phi(x) = \alpha\; a \int \frac{dx}{f(x) + a'x}, \quad \psi(x) = \phi(0) + \phi\left(\frac{x}{a}\right)$$

$$= \alpha a \int \frac{dx}{a\,f\left(\frac{x}{a}\right) + a'x} + \phi(0),$$ where $f(x)$ depends

on the equation $a^{2n} = \{f(x) - nx\}^{n+a'} \{f(x) + nx\}.^{n-a'}$

(X)

"Remarques sur quelques propriétés générales d' une certaine sorte de fonctions transcendantes" (*Crelle's J.*, Bd. 3, 1828; *Oeuvres*, t. 1, pp. 444—456). This treats of Abel's theorem for the particular case of $\int r dx \Big/ \sqrt{R}$ where R is an arbitrary rational and integral function and r is any rational function; the general case is considered in the Paris memoir.

(XI)

"Sur le nombre des transformations différentes qu' on peut faire subir á une fonction elliptique par la substitution d' une fonction rationnelle dont le degré est un nombre premiere donné" (*Crelle's J.*, Bd. 3, 1828; *Oeuvres*, t. 1, pp. 457—465).

Jacobi had not up to the time been able to solve the question considered in this paper.

(XII)

"Note sur quelques formules elliptiques" (*Crelle's J.*, Bd. 4, 1829; *Oeuvres*, t. 1, pp. 467—477).

Abel says: "In the 2nd volume of this journal I have given several formulae for the development of the functions $\varphi(a)$, $f(a)$, $F(\alpha)$, in the case where the moduli e and c are real. It shall be easy to deduce analogous formulae for the case where e^2 is a negative quantity"

We have

$$\frac{w}{2} = b\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-c^2\sin^2\theta)}}$$

where

$$b = \sqrt{(1-c^2)}; \text{ and } \frac{\omega}{2} = b\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-b^2\sin^2\theta)}}.$$

Putting

$$\frac{w'}{2} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-c^2\sin^2\theta)}}, \quad \frac{\omega'}{2} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-b^2\sin^2\theta)}}$$

$$r = e^{-\frac{w'}{\omega}},$$

Abel establishes, among others, the following results:—

$$(1) \quad c^{\frac{1}{4}} = \frac{1-r}{1+r} \cdot \frac{1-r^3}{1+r^3} \cdot \frac{1-r^5}{1+r^5} \cdots\cdots,$$

$$b^{\frac{1}{4}} = \sqrt{(2)}.\ r^{\frac{1}{8}}.\ \frac{1+r^2}{1+r} \cdot \frac{1+r^4}{1+r^3} \cdot \frac{1+r^6}{1+r^5} \cdots\cdots$$

$$(2) \quad (1-r)(1-r^2)(1-r^3)\ldots\ldots = \frac{b^{\frac{1}{12}}\, c^{\frac{1}{3}}}{2^{\frac{1}{6}}\, r^{\frac{1}{24}}} \cdot \sqrt{\left(\frac{\omega}{\pi}\right)}$$

"formula due to M. Jacobi (Tome 3, p. 193, where this mathematician presents many other very remarkable and very elegant formulae)"

$$(3)\quad \lambda\left(\frac{\omega'}{\pi}x\right)=\frac{2}{\surd(c)}\cdot q^{1}\sin x.\ \frac{1-2q^2\cos 2x+q^4}{1-2q\cos 2x+q^4}$$

$$\times\frac{1-2q^4\cos 2x+q^8}{1-2q^3\cos 2x+q^6}\cdots\cdots,$$

where $x=\lambda(a)$ when $a=\int_0^x\frac{dx}{\surd(1-x^2)(1-c^2x^2)}$

(4) "One can prove as I have done in the last number of the *Astronomische Nachrichten* that if two real elliptic functions,

$$F(c,\theta)=\int_0^\theta\frac{d\theta}{\surd(1-c^2\sin^2\theta)},$$

$$F(c',\theta')=\int_0^{\theta'}\frac{d\theta'}{\surd(1-c'^2\sin^2\theta')},$$

of which the moduli c, c' are less than 1, are reducible, one to the other by means of an algebraic relation between $\sin\theta$ and $\sin\theta'$, then one can always find 2 integers m and n such that

$$n.\int_0^{\frac{\pi}{2}}\frac{d\theta}{\surd(1-c^2\sin^2\theta)}\cdot\int_0^{\frac{\pi}{2}}\frac{d\theta}{\surd(1-b'^2\sin^2\theta)}$$

$$=m.\int_0^{\frac{\pi}{2}}\frac{d\theta}{\surd(1-b^2\sin^2\theta)}\cdot\int_0^{\frac{\pi}{2}}\frac{d\theta}{\surd(1-c'^2\sin\theta)}\,.$$

If this condition is fulfilled one can always determine $\sin\theta'$ algebraically in terms of $\sin\theta$ such that

$$F(c',\theta')=a\,F(c,\theta)$$

where a is a constant."

In the above $b'=\surd(1-c'^2)$.

(XIII)

"Mèmoire sur une classe particuliére d' équations résoluble algébriquement" (*Crelle's Journal*, Bd. 4, 1829; *Ocuvres*, t. 1, pp. 478—507).

Abel says in the introduction: "In general, I have demonstrated the following theorem: 'If the roots of an equation of any degree whatever are connected among themselves in such a manner that *all* these roots can be expressed rationally by means of one of them, which we will denote by x; if moreover by denoting by $\theta(x)$, $\theta_1(x)$ two other roots whatever, one has

$$\theta\left\{\theta_1(x)\right\}=\theta_1\left\{\theta(x)\right\},$$

then the equation in question shall be always soluble algebraically. In the same manner, if one supposes the equation irreducible and its degree expressed by

$$a_1^{\gamma_1}\,a_2^{\gamma_2}\cdots\,a_w^{\gamma_w}\,,$$

where $a_1, a_2, \ldots a_w$ are different prime numbers, one can bring back the solution of the equation to that of γ_1 equations of degree a_1, of γ_2 equations of degree a_2, of γ_3 equations of degree a_3, etc. After having expounded the theory in general, I shall apply it to circular and elliptic functions".

(XIV)

"Théoremes sur les fonctions elliptiques" (*Crelle's Journal*, Bd, 4 , 1829; *Oeuvres*, t. 1, pp. 508—514).

Abel says : "The formula given by M. Jacobi in the 3rd volume of this Journal on p. 86 can be easily established by the help of a theorem which we proceed to demonstrate".

(XV)

"Démonstration d'une propriété d'une certaine classe de fonctions transcendantes" (*Crelle's J.*, Bd. 4, 1822 ; *Oeuvres*, t. 1, pp. 515—517).

The theorem is enunciated by Abel as follows: Let y be a function of x which satisfies an irreducible equation whatever of the form

$$0=p_0+p_1 y+p_2 y^2+\ldots+p_{n-1} y^{n-1}+y^n \ldots(1),$$

where $p_0, p_1, \ldots p_{n-1}$ are integral functions of the variable x. Let in the same manner

$$0=q_0+q_1 y+q_2 y^2+\ldots+q_{n-1} y^{n-1}\ldots\ldots \qquad (2)$$

be an equation, where $q_0, q_1 \ldots q_{n-1}$ are equally integral functions of x and let us suppose as variable the coefficients of the different powers of x in these functions. We shall designate these coefficients by

$a, a', a'', ..$ In virtue of the two equations (1) and (2), x shall be a function of $a, a', a''...$ and one shall determine the values by eliminating the quantity y. Let us denote by

$$\rho = 0 \qquad (3)$$

the result of the elimination in such a way that ρ shall contain only the variables $x, a, a', a'',...$ Let μ be the degree of the equation in respect to x and denote by $x_1, x_2,...x_\mu$ the μ roots which shall be functions of $a, a', a'',....$ This being put, if one makes

$$\psi(x) = \int f(x, y)dx,$$

where $f(x, y)$ denotes a *rational* function whatever of x and y, I say that the transcendental function shall enjoy the general property, expressed by the equation, $\psi(x_1) + \psi(x_2) + ...\psi(x_\mu) = u + k_1 \log v_1 + k_2 \log v_2 + ... + k_n \log v_n$, $u, v_1, v_2, ..v_n$ being rational functions of $a, a', a'',...$ and $k_1, k_2,..$ being constants."

(XVI)

"Précis d' une théorie des fonctions elliptiques" (*Crelle's Journal*, Bd. 4, 1829; *Oeuvres*, t. 1, pp. 518—617).

This was intended to be the beginning of a comprehensive work on elliptic functions.

Within five days of his arrival at Berlin on the return journey, Abel wrote to his friend Boeck on the 15th January, 1827, that his whole fortune amounted to 14 thalers and on the 20th he wrote to Professor Holmboë for a remittance as a loan which he got after some delay. Thanking Holmboë for the sum of 293 marks sent to him, Abel says in his letter, dated the 4th March, that by his generosity Holmboë had rendered him a great service as he had "become poorer than a church rat." This small sum helped Abel to stay at Berlin for some months, but when he arrived at Christiania in May, 1827, he was in extreme poverty.

Abel had, ever since 1820 or 1821, when he first became aware of his being a mathematical genius, been fondly hoping to get a University appointment at Christiania as soon as his worth was recognized by experts. Hence his great anxiety to go abroad. His first disappointment came with his failure to get the post at the University rendered vacant by the retirement of Rasmussen. The following quotation from the letter of the Senate or Academic Council of the University sent on the 16th December, 1825, to the Vice-Chancellor will explain matters: "In conformity with the order, dated the 28th ultimo, emanating from the department of ecclesiastic affairs and inviting the Senate to propose a man for appointment to the Chair of Mathematics, vacant in the University, the Senate, in accordance with the opinion given by the Faculty of Philosophy, has the honour to invite attention to two men each of whom is by his learning entirely fit to occupy the post, viz. the teacher of the high school at Christiania, Bernt Michael Holmboë, and the student Niels Abel.

"The first, during the 8 years he has served as the teacher of Mathematics at the high school, has given proof every day of a remarkable capacity, and, at the same time, he has shown, by the writings which he has published, that he possesses an extensive and profound knowledge of Mathematics. Moreover, having served for ten years as amanuensis to Professor Hansteen, he is advantageously known in the University for the solidity of his knowledge and for his good habits.

"The Senate has had equally the opportunity to know the rare talents of the student Abel for Mathematics and his great progress in that science, by the five years of his study here as well as by the memoirs which he has published. One ought to note that he is actually absent on a tour in foreign countries, which he undertook recently, and that one cannot now ask him to interrupt it without prejudice to the advancement of his studies; and at the same time the Senate believes to be able to conclude from the nature of the spirit of Abel that he will find it difficult to adapt his teaching to the intelligence of young students, and, consequently, he will not be able to teach the elements of Mathematics, which form the principal objective of this chair, with the same fruit-fullness as a more experienced master, so that Abel shall be, on the contrary, preferable for occupying a chair of high Mathematics that one may hope to see founded at our University in course of time. For these reasons, the Senate deems it to be its duty to recommend by preference for the vacant chair the appointment of the teacher Holmboë; but it considers also a duty to observe that, in the interest of Science in general and of our University in particular, it will be desirable not to lose sight of the student Abel"*

After Abel's death there was much controversy in the public press. It was stated that Abel had been for years very successful in training candidates privately for the two University examinations, the *examen artium* and the *examen philosophicum*. Against the Senate, it was also rumoured that Holmboë was willing to retire in favour of his talented pupil but the threat to bring in a foreigner silenced him.

Soon after his return, Abel addressed a letter, dated the 2nd June, 1827, to the Senate asking it to recommend his case for the favourable consideration of the Government. The Senate understood the letter to mean a request for subsidy to continue his studies, and recommended the sum of 200 spd. (=1000 fr.) a year which Abel had been getting before his grand tour. But this recommendation was turned down. Becoming desperate, Abel renewed his request and wrote: "I am absolutely without pecuniary resources for procuring for myself lodging and the barest necessaries, and have been so since my return".† With this application Abel sent several issues of *Crelle's Journal* containing his papers. The Governmental reply was again in the negative but it was coupled with the authorization that the University should advance 200 spd. out of its own funds if it thought proper. But of that sum, Abel received only a part. To cut short this tale of misery, we may say that, but for the continued help of Holmboë in whose house Abel found shelter, Abel would not have been able to survive the privations and the family worries. His family at Gjerrestad was in deepest misery; his elder brother was a declared idiot, the next brother after him, still a student, appeared to be dissipated; only his sister, Elizabeth , was earning something as governess in a well-to-do family.

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* See C.-A. Bjerknes's *Niels—Henrik Abel: Tableau de sa vie et de son action scientifique*, 1885.

† Pesloüan, l. c., p. 90.

At about the end of 1827, Abel's sky showed a silver lining, He got the balance of the subsidy due from the University and also earned something from private tuition. "It is touching to see his note-books containing in the midst of transcendental researches small excercises in trigonometry meant for his pupils". At about the beginning of 1828, Professor Hansteen was to go on deputation to Siberia and he proposed that Abel should work as a lecturer at the University in his place and share with Holmboë the discharge of the duties of Hansteen at the military school. The second proposal was immediately accepted and, as regards the first, Abel was at first allowed to lecture at the University without being given the title of Lecturer. Professor Michel Schjelderup, a great philanthrophist, received Abel in his family like a son and Abel was also appointed a member of the Academy of Sciences of Trontdjem. With Keilhau and Möller, who had been his companions on his grand tour, Abel founded a small scientific society. In March, 1828, by a royal decree he got the appointment of Lecturer with a salary of 400 s. p. d. a year. From the millitary school he was getting 11 s. p. d. a month. These sums together with his subsidy would have put Abel in material security, but his family responsibilities meant much expenditure and he was heavily in debt. In August 1828, writing to Madame Hansteen, he says: "I am as poor as a church rat, I have no more than 1 s. p. d."

Leavig the tale of Abel's material troubles, we proceed to give here an account of his researches after his return from the grand tour. The following papers belong to this period:

(1) "Recherches sur les fonctions elliptiques"

(*Crelle's J.*, 1827, 1828; *Oeuvres* t.1, pp. 263—388)

The second part was sent in February. Then, after seeing two enunciations by Jacobi (without demonstration) in the *Astronomische Nachrichten* of 13th June and 2nd Aug., 1827, Abel sent in the *Addition* (pp. 384—388 of *Oeuvres, t.* 1).

(2) "Mémoire sur une classe particuliére d' équations resolubles algébriquement" (*C. J.*, 1829; *Oeuvres*, t. 1, pp. 478—514). This is dated the 29th March, 1828.

(3) "Solution d' un probléme général concernant la transformation des fonctions elliptiques".

(*Astronomische Nachrichten*, 1828)

This memoir was written after Abel had seen in April

1828 in *Astronomische Nachrichten* the proof of Jacobi's two theorems announced earlier. The memoir was finished on the 27th May, 1828, and given to Hansteen. The addition to this memoir was sent on the 25th September, 1828.

(4) "Démonstration d' une propriété genéralé d' une certaine classe de fonctions transcendantes" (*Oeuvres*, t. I, pp. 515–517).

This was sent on 6-1-1829 for publication.

Abel arrived on the 12th December at Froland. From the 6th January, 1829, he was noticed to be keeping to his rooms with a bad cold and bronchitis. On the 8th January he wanted to leave Froland, but was advised not to do so as he brought out blood from his mouth. From the 15th, the condition of Abel became visibly serious and after some days he began to get high fever. The conviction came to him that the medical men who had at Paris thought him to be pthisical were after all right. While delirious, he would sometimes shout: "I will fight for my life, I have no pthisis, the Paris physicians were wrong." The case of Abel grew worse and worse and he died during the early morning hours of the 6th April, 1829.

To judge the claim of priority advanced by the admirers of Abel in the discovery of the various properties of elliptic functions, the two memoirs which he communicated to the *Astronomische Nachrichten* in 1828 and 1829 should be reviewed here.

(XVII) and (XVIII)

"Solution d' un probléme général concernant la transformation des fonctions elliptiques" (*Astronomische Nachrichten*, Bd. 6, No. 138, 1828; *Oeuvres*, t. 1, pp. 403–428).

"Addition au mémoire precedent*" (*Astronomische Nachrichten*, Bd. 7, No. 147, 1829; *Oeuvres*, t. 1, pp. 429–443).

The memoir (XVII) "contains the complete solution of the problem of finding all the rational transformations, although the investigation relating to the case of even degrees is not pursued in all its detail. By contrast, some general propositions of the theory of transformation among others on complex multiplication, are added without proof; Abel enunciates here for the first time that all the singular moduli can be expressed by radicals. The manner in which he attacks the problem well characterizes the penetrating sagacity of Abel. He supposes that the variable y of the new integral is a rational function of the variable x of the primitive integral; there exists then an algebraic equation between the two variables. He finds by a deduction, astonishingly simple, the relation between two roots whatever of this equation; if a root x is equal to $\lambda(\theta)$, where λ denotes the primitive elliptic function, every other root is given by $\lambda(\theta + \text{constant})$; he starts then from the periodicity as the means of determining the constant. He finds thus the rational function y, barring four constants, and then simply shows that they can be chosen in such a manner as to realize the transformation. This is the first time that the double periodicity appears as the means of fundamental research as one applied it later.

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* This is dated from Christiania, the 25th September, 1828.

"One can see from the letters of Abel that he was satisfied with his work which, besides, provoked an exclamation of sincere admiration on the part of Jacobi.* Bessel also, in a letter to Gauss, calls it a chef d' oeuvre. But on the 25th September Abel addressed to the *Nachrichten* of Schumacher a new memoir† on the subject which was inserted in No. 147 in November, 1828. In this the problem is generalized in a manner to comprehend all *algebraic* transformations, the modulus being always supposed real.

"The double periodicity is applied here to establish necessary conditions relating to *the ratios of the periods*, after which it is proved by the expansion in infinite products that these conditions are sufficient. At the same time, Abel affirms that the results of the first memoir are still valid when the modulus is imaginary.

"By these two memoirs the theory of the transformation of the integral of the first kind attained to such a high degree of perfection that since then but little has been added, excluding the most particular researches on the complex multiplication. On the same subject appeared a short note in *Crelle's Journal**, where Abel declares that the transformation of the integrals of the second and the third kinds is a consequence of the transformation of the corresponding integrals of the first kind ; this note was published on the 3rd December."†

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* In a letter to Legendre, Jacobi says that it was "au dessus de ses eloges" as well as "au dessus de ses forces.'

† This is (XVIII).

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* *Oeuvres*, t. 1, p. 466

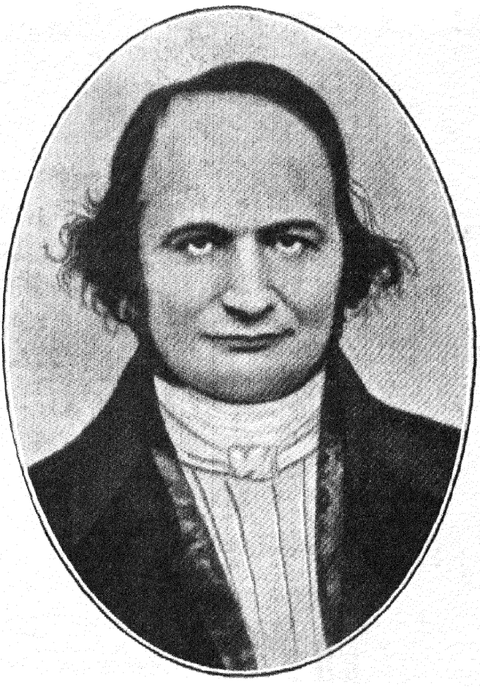
† See L. Sylow's "Les Etudes d' Abel et ses decouvertes"(pp. 47—48).

CHAPTER IV.

JACOBI

Coming from a race, which, although scattered over the face of Europe, has produced distinguished men in almost every sphere of human endeavour, Carl Gustav Jacob Jacobi was born on the 10th December, 1804, at Potsdam near Berlin as the second son of the Jew merchant and banker, Simon Jacobi, and his wife of the family of Lehmann. Of the two brothers of Jacobi, the elder, Moritz, became later a famous physicist of St. Petersburg and the inventor of the science of Galvano-Plastik. The other brother was Eduard, and Jacobi had an only sister, Therese.

The earliest instruction in the classical languages and the elements of Mathematics was received by Jacobi at home from his maternal uncle, Lehmann, who was so successful that the boy was able to take admission in November, 1816, into the second class of the Gymnasium at Potsdam and to be promoted to the first class in six months. He had to remain for four years in that class before appearing at the school final or maturity examination, as, according to the rules for admission into Prussian Universities, it was not possible for any one to receive admission before



JACOBI

(1804—1851)

he was 16 years of age. Dirichlet says of these four years: "The mathematical teaching which was entirely treated as a matter of memory, could not satisfy this young student of the highest class. His relations with the teacher were, therefore, for a long time very unpleasant; but they became better finally, since the teacher had sufficient foresight to let the extraordinary boy do as he liked and to allow him to busy himself with Euler's *Introductio* whilst the remaining boys repeated painfully elementary theorems which they had learnt by heart. How far Jacobi's mental development had proceeded at that time is evidenced by the attempt which he made about that time to solve the general equation of the 5th degree".

In Easter, 1821, Jacobi appeared at his maturity examination in the mathematical part of which he had to treat a problem of spherical astronomy which he solved in a clear and elegant manner ending with the remark "whether this proof appears in any text-book, I do not know, because of my ignorance of the mathematical literature". The leaving certificate of the Gymnasium ran as follows: "Endowed by God with rare mental talents, his knowledge of the languages as well as of Mathematics is as thorough as distinguished, quite extraordinary in Greek and History". As the consistorium thought the praise given to Jacobi by his school teachers to be too high, the teachers repeated proudly that "he is a universal head, possesses extraordinary abilities and a high order of peace of mind; he grasps and comprehends all, without being interrupted by fatigue; at present he studies indeed both Philology and Mathematics, but he will find it difficult to concentrate always on both of them and must at some time make his choice of one".

Jacobi joined the Berlin University in 1821 and for some time attended to philosophical, philological and mathematical studies. In philology his progress was so marked that he was specially noted by the famous Böckh who was then the head of the Philological Institute. But in course of 2 year's time, Jacobi decided to concentrate his efforts on mathematical studies alone. Now, in the first two decades of the 19th century, leaving Gauss, there was no man at any German University who was either a great teacher or an eminent researcher. No doubt, Gauss was admired by all who loved research, but he was a failure as a teacher for lack of that "external and manifest warmth and enthusiasm" which the teacher of so dry and positive a science as Mathematics must possess. Jacobi appears to have attended very few mathematical lectures at the University, as they were of too elementary a character to benefit one who had already mastered the chief works of Euler and Lagrange. But "for that very reason he looked more eagerly into the mathematical literature and sought to have a general view of the great store of scientific literature contained in the various academic collections". He wrote to his uncle at the time of his decision to devote himself to Mathematics: "The huge structure which has been raised by the efforts of an Euler, a Lagrange and a Laplace, requires the vastest force and intensity of meditation if one wishes to dig deep into its inner nature and not only to scratch it superficially. In order to be master over that structure, so that one may not fear to be crushed, by it, requires a strain which does not allow one to rest until one stands at the top and overlooks the whole. Then alone, it is also possible to work properly and with composure at the completion of each single part and to extend further the whole great structure when one has grasped its spirit".

In the autumn of 1825, Jacobi obtained the degree of Ph. D. with his thesis, "Disquisitiones analyticae de fractionibus simplicibus" which deals with the resolution of algebraic fractions. Lagrange's formula is :—

$$f(z) = \Sigma u_\lambda \cdot \frac{g(z)}{(z-z_\lambda)g'(z_\lambda)}, \quad (\lambda=0, 1, \ldots n),$$

where

$$g(z) = (z-z_0)(z-z_1)\ldots(z-z_n),$$
$$f(z)=u_\lambda \text{ for } z=z_\lambda,$$

and the quantities λ_0. λ_1,... λ_n are all different. Jacobi extends the result to the case of equal values of z_λ.

Malfatti and Lagrange had worked out the extension by first taking the zeros of $g(z)$ to be different but very little different from one another and then proceeding to the limit. Jacobi obtains a theorem of Lagrange by using the result

$$\frac{1}{(m-a)!}\,\frac{d^{m-1}}{da^{m-1}}\left(\frac{\Pi(a)}{x-a}\right)=\left[\frac{\Pi(a+h)}{x-a-h}\right]_{h=m-1}$$

and thence immediately derives the expression in partial fractions of a rational function without any restriction on the order of its infinity.

Amongst the results given by Jacobi may be mentioned :

$$\frac{x^a}{(x-a_1)\dots(x-a_n)}=\Sigma\,\frac{a_1{}^a\,a_2{}^{n-2}}{(x-a_1)\,(x-a_2)}\cdot\frac{1}{M_{12}},$$

where

$$M_{12}=(a_1-a_3)\dots(a_1-a_n)\ (a_2-a_3)\dots(a_2-a_n),$$

and the summation extends to the a's and a has the values 0, 1, 2,...$n-2$.

(2) The sum of the series

$$\frac{1}{a}+\frac{x(a+c)}{a(a+1)}+\frac{x^2(a+c)\,(a+c+1)}{a(a+1)\,(a+2)}+\dots\text{to inf.}$$

$$=\frac{1}{(1-x)^c x^a}\int x^{a-1}\,(1-x)^{c-1}\,dx.$$

The dissertation concludes with investigations about the transformation of series.

In his copy of the dissertation found after the death of Jacobi there are in his own hand many alterations of style, abbreviations and also some important additions which were published by Weierstrass in 1884. Among these may be mentioned the following interesting result:—

If $\phi(x) = (x-a_1)\,(x-a_2)\,\dots\,(x-a_n)$, and $f(x)$ and $\chi(x)$ are arbitrary integral functions, then the sum

$$\Sigma\frac{(a_1-a_2)^2\{f(a_1)\chi(a_2)+f(a_2)\chi(a_1)\}}{\phi'(a_1)\phi'(a_2)(\chi-a_1)(\chi-a_2)}$$

will differ from

$$\left[\frac{(x-y)\{f(x)\chi(y)+\chi(x)f(y)}{\phi(x)\phi(y)}\right]_{y=-1}$$

by an integral function of x.

I proceed now to describe as briefly as possible each of the next 21 publications of Jacobi ending with the *Fundamenta nova theoriae functionum ellipticarum* which appeared in April, 1829 :—

I. "Ueber Gauss neue Methode, die Werthe der Integrale naherungsweise zu finden" (*Crelle's Journal*, Bd. I., 1826, pp. 301—308). The introduction to this paper is so explanatory that it may be quoted in extenso: "In the *Principia* of Newton one reads a method how to plot a parabolic curve through a number of given points. This problem appears analytically as an interpolation problem to find the general term of a series from several terms. It is the known case, if the intervals of the ordinates of the given points are equally great, or, expressed analytically, if the values of the succeeding elements, for which also the values of the corresponding members of the series are given, form an arithmetical progression. However, the elegant Algorithm, which Newton gave and which is without justification very little known, stretches itself already to the general case if those intervals of the ordinates of the given points or those values of the elements in series are arbitrary. Newton has made of this an application to quadrature. Through several points, of the curve whose quadrature is sought, for which the ordinates have been calculated, he takes the parabolic curve, and the quadrature of that curve between the same limits as those between which the quadrature of the given curve ought to be found, gives an approximate value.

"Newton has further treated of that interpolation-problem and its application to the quadrature in a booklet which is entitled *Methodus Differentialis* and was first found attached with other memoirs to the Amsterdam edition of his *Principia* of the year 1723. Here he advises among other things the framing of tables, for the sake of the easier calculation of the integrals, for each number of the calculated ordinates whose intervals he takes to be equal, and of such tables he himself makes a beginning which later Roger Cotes has continued in his *harmonia mensurarum.*

"However, Gauss has shown in the Göttingen Commentaries that, by suitable choice of the abscissae, for which the ordinates are calculated, one can double the degree of approximation; and, since such determination occurs independently of the nature of the curve whose quadrature is required, it is possible also according to the method, perfected as shown, to supply tables of which also Gauss gives a sample. Gauss attains to his results by the way of a difficult induction which, by the so-called Kästner's method, if something holds for number n to prove the same for $n+1$, can lead to generality. Therefore a direct proof is still a desideratum. The great simplicity and elegance of the Gaussian results, admits of presuming the existence of a simple way. The object of this memoir is to attain to those results, with which Gauss has enriched science, by such a simple and direct way."

(II)

"Ueber den Ausdruck der verschiedenen Wurzeln einer Gleichung durch bestimmte Integrale" (*Crelle's Journal*, Bd. 2, 1827, pp. 1—8).

Jacobi begins the paper as follows: "In the first volume of the *Mémoires des savans étrangers* of 1806, Parseval has attempted to sum by means of definite integrals the series of Lagrange which express the roots of a given equation, and to prove in a direct manner the formula thus found by him, Later Bessel made in the memoirs of the Berlin Academy of the year 1816—17 a famous application of definite integrals to the problem of Kepler. Since recently the remarkable formulae which Cauchy has given in the 19th part of the polytechnical journal have again drawn the attention of analysts to the determination of the different roots of an equation in this manner, therefore it may be permitted for me to take up again the investigations begun by Perseval and to complete them."

(III)

"Ueber eine besondre Gattung algebraischer Functionen die aus der Entwicklung der Function

$$(1-2xz+z^2)^{-\frac{1}{2}}$$

entstehen" (*Crelle's Journal,* Bd. 2, pp. 223–226; the paper is dated August, 1826).

Jacobi points out (1) that $P_n(x)$ which he denotes by $X^{(n)}$ equals

$$\frac{1}{2^n.\,n!}\,\frac{d^n}{dx^n}\,(x^2-1)^n,$$

a property overlooked by Legendre, the discoverer of P_n, and (2) that

$$\frac{d^{n-r}}{(n-r)!\,dx^{n-r}}\,(x^2-1)^n=(x^2-1)^r\,\frac{d^{n+r}\,(x^2-1)^n}{dx^{n+r}.\,(n+r)!},\quad r\leq n.$$

(IV)

"Ueber die Hauptaxen der Flächen der zweiten Ordnung" (*Crelle's Journal,* Bd. 2, 1827, pp. 227–233; the paper is dated May, 1827).

The introduction runs as follows: "The problem of referring a surface of the second order to its principal axes, requires, as is well-known, the transformation of an expression of the form $Ax^2+By^2+Cz^2+2\,ayz+2bzx+2cxy$, where x, y, z denote the co-ordinates of a point, by the introduction of a new system of rectangular co-ordinates into the form $L\xi^2+Mv^2+N\delta^2$. I shall suppose in what follows that the original system of co-ordinates in reference to which the equation of the surface is given is an oblique one. The problem, conceived in this generality, includes both the cases, viz. where the original co-ordinate system is rectangular or an oblique but conjugate, which both have been treated before."

(V)

First letter to Schumacher, (*Astronomische Nachrichten*, 1827; the letter is dated the 13 th June, 1827).

Second letter to Shumacher, (*Astronomische Nachrichten*, 1827; the letter is dated the 2nd of Aug. , 1827).

[Both appeared on pp. 33-38 in Sept., 1827]

In the first letter Jacobi says: "Be good enough, Monsieur, to insert in your journal the notes on the elliptic transcendentals which I have the honour to address to you. In those notes I flatter myself to have made some discoveries sufficiently interesting in this theory for the exposition to be submitted to the judgment of mathematicians. The integrals of the form

$$\int \frac{d\phi}{\sqrt{(1-c^2 \sin^2\phi)}}$$

appertain according to the diversity of the modulus *c* to different transcendentals. One knows only a single system of moduli such that one may reduce one (of these moduli) to another, and Mr. Legendre says in his *Exercises* that there is only this single system. But in fact there are as many of these systems as there are prime numbers, that is to say, there are an infinite number of these systems each independent of the others, of which any corresponds to a prime number and the known system corresponds to the prime number 2".

As explained by Koenigsberger, "Jacobi announces, although without proof, the general theorem of the rational transformation of the elliptic integrals of the first kind, viz. that sin ϕ, put equal to $\frac{u}{v}$, where u denotes a certain odd function of the nth degree in sin ψ, v a definite even function of degree $n-1$ in sin ψ, furnishes the relation

$$\int \frac{d\phi}{\sqrt{(1-c^2 \sin^2 \phi)}} = m \int \frac{d\psi}{\sqrt{(1-k^2 \sin^2 \psi)}},$$

and points out that one can in almost analogous manner express sin ψ through sin θ in such a manner that by combining the two integral equations one obtains

$$\int \frac{d\phi}{\sqrt{(1-c^2 \sin^2 \phi)}} = n \int \frac{d\theta}{\sqrt{(1-c^2 \sin^2 \theta)}},$$

where sin ϕ is representable by a rational fraction in sin θ of which the numerator contains the odd powers up to the n^2th and the denominator the even powers up to the (n^2-1) th. The theorem, given without proof and without the analytical expression for the general transformation, is further worked out for the transformations of the 3rd and 5th degrees and connected with the multiplication and division corresponding to these numbers." With reference this, Jacobi says: "Thus I give here for the first time the algebraic solution of the equation of the 9th degree on which the trisection of our transcendental depends." Apparently Jacobi was ignorant of the fact that the transformation of the 3rd degree had been already given by Legendre in 1827 in his *Traité des fonctions elliptiques.*

In the second letter, Jacobi actually gives, although without proof, the analytical expression for the general transformation and indeed in the form $\tan\left(45^\circ - \frac{\psi}{2}\right)$

$$= \frac{\tan\frac{\phi' - \phi}{2} . \tan\frac{\phi''' + \phi}{2} . . . \tan\frac{\phi^{(p-2)} \pm \phi}{2}}{\tan\frac{\phi' + \phi}{2} . \tan\frac{\phi''' - \phi}{2} . . . . \tan\frac{\phi^{(p-2)} + \phi}{2}} . \tan\left(45^\circ \mp \frac{\phi}{2}\right)$$

wherein the amplitudes $\phi^{(m)}$ are defined by the relation

$$F(k, \phi^{(m)}) = \frac{m}{p} F(k, 90^\circ)$$

and p denotes an arbitrary odd number.

(VI)

"De residuis cubicis commentatio numerosa" (*Crelle's Journal*, Bd. 2, 1827, pp. 66–69; dated the 22nd June, 1827.)

About this paper, Dirichlet says:—"The paper on cubic residues contains only theorems without proofs. But these theorems are such that they cannot be the result of pure induction. They leave no doubt on this that Jacobi must have been already at that time in possession of new fertile principles in relation to the scientific field which Gauss had opened a quarter of a century earlier to mathematical speculation and which belongs as much to Higher Algebra as to the theory of Numbers. The theorems referred to above are the following:—

1. If p and q are two prime numbers of the form $6n+1$, $4p=L^2+27M^2$, $x^2+3\equiv 0 \pmod q$, then q is then and only then a cubic residue of p when

$$\frac{p(L+3Mx)}{2} \text{ or } \frac{L+3Mx}{L-3Mx}$$

is a cubic residue of q.

2. If the prime number p is of the form $6n+1$, $4p=L^2+27M^2$, and further q is of the form $6n-1$, then q is then and only then a cubic residue of p when

$$\frac{L+M\sqrt{-3}}{L-M\sqrt{-3}}$$

is a cubic residue of q.

Jacobi also generalizes a theorem of Gauss.

(VII)

"Euleri formulae de transformatione coordinatarum" (*Crelle's Journal*, Bd. 2, 1827, pp. 188—189; the paper is dated June, 1827).

In this short note, Jacobi reproduces only the formulae of Euler and Lexell, who had found, for the cœfficients of the coordinates—transformation, expressions by means of the three angles ϕ, ψ, θ which come under consideration in the treatment of mechanical problems.

(VIII)

"De singulari quadam duplicis integralis transformatione" (*Crelle's Journal*, Bd. 2, 1827; the paper is dated June, 1827).

In this paper which, according to Koenigsberger, formed the beginnings of his investigations on elliptic integrals and functions, Jacobi shows that if

$$e=a+a'\cos^2\psi+a''\sin^2\psi\cdot\cos^2\phi+a'''\sin^2\psi\sin^2\phi+2b'\cos\psi+2b''\sin\psi\cos\phi+2b'''\sin\psi\sin\phi+2c'\sin^2\psi\cos\phi\times\sin\phi+2c''\cos\psi\sin\psi\sin\phi+2c'''\cos\psi\sin\psi\cos\phi,$$

then the double integral

$$\int\int\frac{\sin\psi d\psi d\phi}{e}$$

admits of being transformed into

$$\int\int\frac{\sin PdPdv}{G+G'\cos^2P+G''\sin^2P\cos^2v+G'''\sin^2P\sin^2v}$$

by means of expressions for cos P, sin P cos v, sin P $\times$ sin v, which represent linear broken functions of cos ψ, sin ψ cos ϕ, sin ψ sin ϕ.

(IX)

"Ueber die Integration der partiellen Differentialgleichungen erster Ordnung" (*Crelle's Journal*, Bd. 2, 1827, pp. 317—319; the paper is dated the 12th August, 1827).

In the introduction to his paper, Jacobi begins by saying that the origin and building up of the theory of partial differential equations forms an event of the greatest importance in the history of Mathematics in the 18th century. After mentioning briefly the contributions of Euler and Lagrange, he says: "Pfaff considered the subject from a point of view entirely different from any previous one. He looked upon the whole problem of the integration of partial differential equations of the first order as a special case of the very comprehensive problem, to integrate every ordinary linear differential equation in $2n$ variables by means of a system of n equations; and whilst he solved this problem completely, he completed at the same time the theory of partial differential equations of the 1st order.

"In the following I shall take up the subject again and subject it to what I believe to be a new treatment which will perhaps recommend itself to analysts because of its being easy; by means of it the difficulties which stood against the extension of Lagrange's method to any number of variables, so far as the nature of this method makes it possible, are removed, and one thus obtains by an opposite way the same results as Pfaff had obtained".

(X)

"Ueber die Pfaffsche Methode, eine gewöhnliche lineare Differentialgleichung zwischen $2n$ Variabeln durch ein System von n Gleichungen zu integriren" (*Crelle's Journal*, Bd. 2, 1827, pp. 347—357; the paper is dated the 14th August, 1827).

In the previous paper Jacobi had said in conclusion that he "would treat Pfaff's method in several papers." The present paper is the first of such papers and is really "only a clear and somewhat extended description of Pfaff's method".* In an addendum to the paper,

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* Koenigsberger: *Carl Gustav Jacob Jacobi, Festschrift*, 1904, p. 48

Jacobi says:—"After the completion of this paper I noted that the equations at which Lagrange and Poisson have arrived in their famous works on the variation of constants in the problems of Mechanics likewise form such a system as we have examined here. Let one see the 15th part of the polytechnical journal, pages 288—289. Since the method of Pfaff likewise rests on the variation of constants, so this system of equations appears to be prominent in the method of variation of constants." Koenigsberger* recognizes in this addendum a promise of the epoch-making investigations in Mechanics which Jacobi carried out much later.

(XI)

"Ueber die Bestimmung der Rectascension and Declination eines Sternes aus den gemessenen Distanzen dessellben von zwei bekannten Sternen" (*Crelle's Journal*, Bd. 2, 1827, pp. 345—346; the paper is dated the 20th Aug. 1827).

"The works of Bessel and oral examinations with him of astronomical questions led Jacobi" to this small paper "in which are derived, in a short and direct way without the help of Spherical Trigonometry, formulae, which are partly known but complicated and not suited to an elegant method of calculation."†

(XII)

Demonstration of the two theorems enunciated in

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* l. c., p. 50.

† Koenigsberger, l. c., p. 50.

June and August, 1827. *Astronomischen Nachrichten*, Vol. VI, 18th Nov., 1827).

(XIII)

"Addition au Mémoire de M. Abel sur les fonctions elliptiques, insere dans le vol. II. de ce journal, cah 2, pag. 101" (*Crelle's Journal* Bd. 3, 1828, p. 86, the paper is dated the 25th Jan., 1828).

In this paper of one page, Jacobi says: "M. Abel, in his excellent memoir on the elliptic functions, has been the first to prove that the equations of the n^2 th degree on which the division of an elliptic function of the first species in n parts depends, can be solved algebraically However, the method of that author is susceptible of a great simplification. I propose to show this in two words while using the notation of M. Abel." The paper of Abel referred to is his *Recherches*.

(XIV)

"Ueber die Anwendung der elliptischen Transcendenten auf ein bekanntes Problem der Elementargeometrie: Die Relation zwischen der Distanz der Mittelpuncte und den Radien zweier Kreise zu finden, von denen der eine einem unregelmaessigen Polygon eingeschrieben, der andere demselben umgeschrieben ist" (*Crelle's Journal*, Bd. 3, 1828, pp. 376—389; the date of the paper is 1st April, 1828).

"Jacobi develops in this paper a remarkable connection between the well-known Poncelet's closure problem and the theory of elliptic functions. In fact let one draw from a point A of a circle with the centre C and the radius R a tangent to an inner circle with the centre c and the radius r, and lay down further from the point of intersection A′ of that tangent with the first circle again a tangent to the inner circle up to A″, and so on; finally let one prolong $cC=a$ up to the point of intersection P with the first circle and call the angle $ACP=2\phi$, $A'CP=2\phi'$, $A''CP=2\phi''$, and so on; then Jacobi finds the relations

$$\tan\frac{\phi^{(r+1)}+\phi^{(r-1)}}{2}=\frac{R-a}{R+a}\tan\phi^{(r)}$$

which agree with those which are used for the multiplication of the argument t of the elliptic function am $(u+t)$ and deduces thence easily the theorem: If one determines the quantities α and k from the equations

$$\cos\alpha=\frac{r}{R+a},\quad k^2=\frac{4Ra}{(R+a)^2-r^2},$$

then the relation, which holds between R, r and the distance a of the two centres, if to the first circle an n-ogon is circumscribed which is inscribed to the second and h denotes the number of the courses of the polygon through the whole periphery, is given in Legendre's notation in the simple form

$$F(\alpha, k)=\frac{h}{n}\;F(\pi, k)^{\prime\prime}.\dagger$$

(XV)

"Note sur les fonctions elliptiques" (*Crelle's Journal*, Bd. 3, 1828, pp., 192—195; this is from a letter dated the 2nd April, 1828).

This very short but fundamental paper contains esults of two types. *First*, we have the formulae

$$(1)\ \operatorname{sn}\frac{2Kx}{\pi}=\frac{2}{\sqrt{k}}\,\frac{\sum\limits_{0}^{\infty}(-1)^n.\,q^{(2n+1)^2/4}\sin(2n+1)x}{1+2\sum\limits_{1}^{\infty}(-1)^n q^{n^2}\cos 2nx}$$

with similar expressions for cn $\frac{2Kx}{\pi}$, $dn\,\frac{2Kx}{\pi}$ and

$$\sqrt{\left(\frac{1-\operatorname{cn}\theta}{1+\operatorname{cn}\theta}\right)},\quad \theta \text{ being } \frac{2Kx}{\pi};$$

$$(2)\ \left(\frac{2K}{\pi}\right)^{\frac{1}{2}}=1+2\sum_{1}^{\infty}q^{n^2}$$

$$\text{and } \frac{1}{2}\sqrt{\left(\frac{2k}{\pi}K\right)}=\sum_{1}^{\infty}q^{\frac{(2n-1)^2}{4}}$$

where $q=e^{-\pi K'/K}$.

Jacobi says: "In the present state of the subject one can say that a series is summed if it has been reduced to elliptic functions. Thus he gives $\left(\frac{k^2}{q}\right)^{\frac{1}{24}}.\,(2k')^{\frac{1}{3}}\times\left(\frac{K}{\pi}\right)^{\frac{1}{2}}$ as the sum of Euler's series $1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+\ldots$ which is equal to the infinite product

$$(1-q)\,(1-q^2)\,(1-q^3)\ldots\ldots$$

Secondly, Jacobi gives results relating to the transformation of one modulus into another. For example, he says that, n being a prime number, by means of a substitution an elliptic integral with a modulus k can be transformed into another with modulus λ, λ having $n+1$ different values, viz. those corresponding to

$$q^n,\ q^{\frac{1}{n}},\ a\,q^{\frac{1}{n}},\ a^2\,q^{\frac{1}{n}}, \ldots a^{n-1}\,q^{\frac{1}{n}}$$

in the place of q, where $a^n=1$. Only two of these moduli are real: they are given by

$$\tfrac{1}{2}\sqrt{(\lambda)}=\left\{\sum_{m=1}^{\infty} q^{n.\frac{(2m-1^2)}{4}}\right\}\Big/\left\{1+2\sum_{1}^{\infty} q^{n.m^2}\right\},$$

$$\tfrac{1}{2}\sqrt{(\lambda)}=\left\{\sum_{m=1}^{\infty} q^{\frac{(2m-1)^2}{4n}}\right\}\Big/\left\{1+2\sum_{1}^{\infty} q^{\frac{m^2}{n}}\right\}.$$

Again there is the differential equation

$$3\left(\frac{d^2\lambda}{dk^2}\right)^2-2\ \frac{d\lambda}{ak}\ \frac{d^3\lambda}{dk^3}+\left(\frac{d\lambda}{dk}\right)^2\times$$

$$\left\{\left(\frac{1+k^2}{k-k^3}\right)^2-\frac{1+\lambda^2}{\lambda-\lambda^3}\left(\frac{d\lambda}{dk}\right)^2\right\}=0$$

as the condition for the transformability of λ into k.

Jacobi points out that the above equation admits of an infinite number of algebraic solutions.

For example, putting $k^{\frac{1}{4}}=u$, $\lambda^{\frac{1}{4}}=u$, we have

$$u^4-v^4-2uv\,(1-u^2v^2)=0 \text{ for } n=3;$$

$$u^6-v^6+5u^2v^2\,(u^2-v^2)-4uv\,(1-u^4v^4)=0 \text{ for } n=5.$$

Jacobi also points out that in some cases λ^2 may equal k^2. Thus when $k=\frac{1}{\sqrt{(2)}}$, for $n=5$, the above equation gives $v=(1+i)\,u^5$, *i. e.*, $v^8=\lambda^2=\frac{1}{2}$.

(XVI)

"Note sur la decomposition d' un nombre donnè en quatre carrés" (*C. J.*, Bd. 3, 1828, p. 191; the paper dated the 24th April, 1828)"

In this paper of one page is given by Jacobi without proof the following result: "I deduce from *the theory of elliptic functions* the theorem: Let n be any odd integer whatever. If one can find in m ways four odd numbers a, b, c, d such that

$$a^2+b^2+c^2+d^2=4n,$$

then, by having regard to the accounting as one solution when the squares are permuted among themselves, m shall be equal to the sum of the factors of the number n.

"Therefore if, for example, n is a prime number, one shall have $m=n+1$.

"Let, for example, $n=15$: one has

$$4\cdot 15=60=1+1+3^2+7^2=5^2+5^2+1+3^2.$$

"Because two squares in these two solutions are equal one has $m=12+12=24$."

(XVII)—(XIX)

"Suite des notices sur les fonctions elliptiques" (*Crelles Journal*, Bd. 3, 1828, pp. 303—310; this is taken from a letter, dated the 21st July, 1828).

"Suite des notices sur les fonctions elliptiques" (*Crelle's Journal*, Bd. 3, 1828, pp. 403—404; this is taken from a letter, dated the 3rd October, 1828).

"Suite des notices sur les fonctions elliptiques (*Crelle's Journal*, Bd. 4, 1829, pp. 185-193; this is taken from a letter, dated the 11th January, 1829).

(XVII) The results of this paper may be briefly enumerated under three heads.

(A)

(1) The function $\Theta(x)$ is introduced as an independent transcendental by the definition

$$\Theta(x) = 1 - 2q \cos 2x + 2q^4 \cos 4x - 2q^9 \cos 6x + \dots$$

where

$$-\log q = \frac{\pi K'}{K} = \omega,$$

and

$H(x)$ is defined as

$$iq^{\frac{1}{4}}\, e^{-ix}\, \Theta\left(x - \frac{i\omega}{2}\right)$$

so that

$$H(x) = 2q^{\frac{1}{4}} \sin x - 2q^{\frac{9}{4}} \sin 3x + 2q^{\frac{25}{4}} \sin 5x - \dots$$

(2) The equations

$$\Theta(x+\pi) = \Theta(x), \Theta(x+i\omega) = -\frac{e^{-2ix}}{q}\, \Theta(x),$$

$$H(x+\pi) = -H(x); H\left(x + \frac{i\omega}{2}\right) = \frac{ie^{-ix}}{q^{\frac{1}{4}}}\Theta(x),$$

and other equations similar to the above are given.

(3) The elliptic functions sn $\frac{2Kx}{\pi}$, cn $\frac{2Kx}{\pi}$ and dn $\frac{2Kx}{\pi}$ are expressed respectively as equal to

$$\frac{1}{\surd(k)}\,\frac{H(x)}{\Theta(x)},\ \surd\left(\frac{k'}{k}\right)\frac{H\left(x+\frac{\pi}{2}\right)}{\Theta(x)},\ \surd(k')\,\frac{\Theta\left(x+\frac{\pi}{2}\right)}{\Theta(x)},$$

and the quantities

$\sqrt{\left(\frac{2K}{\pi}\right)}, \sqrt{\left(\frac{2k'K}{\pi}\right)}, \sqrt{\left(\frac{2kK}{\pi}\right)}$ are expressed as equal to $\Theta\left(\frac{\pi}{2}\right)$, $\Theta(0)$, $q^{\frac{1}{4}}\,\Theta\left(\frac{\pi}{2}+\iota\omega\right)$.

(4) $\Theta(x)$ and $H(x)$ are expanded in infinite products and hence we are given*

$$C'\,\Theta(nx, q^n)=\Theta(x)\,\Theta\left(x+\frac{2\pi}{n}\right)\ldots\ldots\Theta\left(x+2\frac{n-1}{n}\pi\right),$$

$$C'\,H(nx, q^n)=(-1)^{\frac{n-1}{2}}H(x)H\left(x+\frac{2\pi}{2}\right)\ldots H\left(x+2\frac{n-1}{n}\pi\right),$$

and then the general transformation formula

$$\operatorname{sn}\left(\frac{2nK^{(n)}x}{\pi}, k^{(n)}\right)$$

$$=(-1)^{\frac{n-1}{2}}\sqrt{\left(\frac{k^n}{k^{(n)}}\right)}\operatorname{sn}\frac{2Kx}{\pi}\ \operatorname{sn}\frac{2K}{\pi}\left(x+\frac{2\pi}{n}\right)\ldots$$

$$\times\operatorname{sn}\frac{2K}{\pi}\left(x+\frac{2(n-1)\pi}{n}\right)$$

where $K^{(n)}$, $k^{(n)}$ are quantities which depend in the same manner on q^n as K, k depend on q.

(5) Among the other results are the partial differental equations

$$\frac{\partial^2\Theta}{\partial x^2}=4\frac{\partial\Theta}{\partial\omega},\quad \frac{\partial^2 H}{\partial x^2}=4\frac{\partial H}{\partial\omega},$$

and the developments of Θ and H in powers of x.

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* C′ is a constant, and n is an odd number.

(B)

The expression of the function $\Theta(x)$ in terms of elliptic integrals is given by the equation

$$\log \frac{\Theta(x)}{\sqrt{\left(\frac{2k'K}{\pi}\right)}} = \int_0^{\phi} \frac{F^1 E(\phi) - E^1 F(\phi)}{F^1 \Delta\psi} d\phi \, ,$$

ϕ being always the amplitude of $\frac{2Kx}{\pi}$.

The elliptic integral of the 3rd kind

$$\int_0^{\phi} \frac{2k^2 \sin A \cos A \Delta A \sin^2\phi \, d\phi}{(1 - k^2 \sin^2 A \sin^2\phi)\Delta\phi} - \frac{2F(\phi)}{F^1} \times$$

$$\left\{F^1 E(A) - E^1 F(A)\right\}$$

is given to be equal to

$$\log \frac{\Theta(x-a)}{\Theta(x+a)} ,$$

where $A = \text{am} \frac{2Ka}{\pi}$.

(C)

Calling M the multiplicator in the equation

$$\frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = \frac{M dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

which is satisfied by a rational expression for y in terms of x with no powers higher than x^n appearing, it is stated that M can be expressed rationally in k and λ by the general formula

$$M^2 = \frac{n\,(k-k^2)}{(\lambda-\lambda^3)}\,\frac{d\lambda}{d\kappa}\,.$$

Eliminating λ we have an algebraic equation between M and k, which has the remarkable property that the square roots of its $(n+1)$ roots can be expressed linearly in terms of other quantities $\frac{n+1}{2}$ in number; here n is an odd prime number.

(XVIII). This paper of two pages gives

$$\Theta\,(i\,x,\,q) = \sqrt{\left(\frac{K}{K'}\right)}\; e^{\frac{Kx^2}{\pi K'}}\; H\left(\frac{Kx}{K'} + \frac{\pi}{2}\,,\,q'\right),$$

with a similar equation for $H\,(i\,x,\,q)$, and indicates how the general expression for the coefficients A's in the transformation $z = \frac{U}{V}$, where

$$U = A'x + A'''x^3 + \ldots + A^{(n^2)}\,x^{n^2},$$

$$V = 1 + A''x^2 + A^{(4)}\,x^4 + \ldots + A^{(n^2-1)}\,x^{n^2-1},$$

can be found by a recurrence formula between $A^{(m)}$, $A^{(m-2}$ and $A^{m-4)}$.

(XIX). The following results of this paper may be emmerated here as of special importance:—

(1) In the transformation $y = \frac{U}{V}$ of the n th degree, both U and V satisfy the equation

$$n(n-1)x^2z + (n-1)(ax-2x^3)\frac{\partial z}{\partial x} + (1-ax^2+x^4)\,\frac{\partial^2 z}{\partial x^2}$$

$$= 2n(a^2-4)\,\frac{\partial z}{\partial a}\,,$$

where $$a=\frac{1+k^2}{k};$$

this equation has algebraical integrals and its general integration reduces to that of

$$\frac{\partial^2 z}{\partial v^2}=\frac{\partial z}{\partial \omega}.$$

(2) Putting

$$Z(u) \text{ for } \frac{F^1\,E(\phi)-E^1\,F(\phi)}{F^1},$$

where $\phi=\text{am}\,(u)$, we have the transformation formula for the elliptic integral of the third kind

$$\Pi\left(\frac{u}{M},\ \frac{a}{M},\ \lambda\right)-n\,\Pi(u,\ a,\ k)$$

$$=\frac{u}{M}\,Z\left(\frac{a}{M},\ \lambda\right)-nuZ\,(a,\ k)$$

$$+\ \tfrac{1}{2}\log\prod_{m=1}^{\frac{n-1}{2}}\frac{1-k^2\,\text{sn}^2\,2m\omega\ \text{sn}^2\,(u-a)}{1-k^2\,\text{sn}^2\,2m\omega\ \text{sn}^2\,(u+a)}.$$

(3) Sn (u,k) is expressed in terms of sn $\left(\frac{u}{M},\lambda\right)$ and sn $\left(\frac{u}{M},\lambda\right)$ in terms of sn $(n\ u,\ k)$ and the simplest expressions are obtained by the algebraic solubility of the division equation of the n th degree.

(4) The paper concludes with the enunciation of the theorem : that being supposed known all the moduli into which one can transform a given modulus by the help of a transformation corresponding to the number n, one can express, without solving any algebraic equation, by these moduli all the quantities of the

form $\mathrm{sn}^2 \dfrac{2m\mathrm{K}+2m' i \mathrm{K}'}{n}$, m, m' being any numbers whatever.

(XX)

"De functionibus ellipticis commentatio." (*C. J.*, Bd. 4, 1829, pp. 371—390; the paper is dated April, 1829.)

This paper was sent out to Crelle immediately after the appearance of the *Fundamenta*. The paper is divided into eleven articles and contains a large number of results in deriving which frequent references have been made to the *Fundamenta*. The function $\Omega(v)$ defined by

$$\log \Omega(u) = \int_0^u \mathrm{E}(u)\,du$$

is studied and many of its properties, including its addition theorem, have been given. In this paper, the way, converse of that adopted before for the derivation of elliptic integrals from the addition theorem of Θ, is struck, viz. with the help of the integrals relations or thetas with composite arguments are obtained.

(XXI)

*Fundamenta Nova Theoriae Functionum Ellipticarum**, Regiomonti 1829. (*Gesammelte Werke*, Bd. 1, pp. 49—239).

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* There are 66 articles, grouped under two heads: De transformatione functionum ellipticarum (Arts. 1—34), and Theoria evolutionis functionum ellipticarum (Arts. 35—66).

Prooemium

Ante biennium fere, cum theoriam functionum ellipticarum accuratius examinare placuit, incidi in quaestiones quasdam gravissimas, quae et theoriae illi novam faciem creare et universam artem analyticam insigniter promovere videbantur. Quibus ad exitum felicem et propter difficultatem rei vix exspectatum perductis, prima earum momenta breviter et sine demonstratione, mox cum vehementius illa desiderari et invento novo vix fides tribui videretur, addita demonstratione, cum geometris communicavi. Urgebar simul, ut systema completum questionum a me susceptarum in publicum ederem. Cui desiderio ut ex parte saltem satisfacerem, fundamenta, quibus quaestiones meae superstructae sunt, in publicum edere constitui. Quae fundamenta nova theoriae functionum ellipticarum iam indulgentiæ geometrarum commendamus.

(*New Foundations of the theory of elliptic functions*, Koenigsberg, 1829. *Gesammelte Werke*, Bd. 1, pp. 49-239).

PREFACE.

When I was engaged, some two years ago, in a more rigorous study of the theory of elliptic functions, I came across certain important investigations, which seemed to give to this theory quite a new aspect, and also to advance in a remarkable way the general theory of mathematical Analysis. I was successful in my researches, although I hardly expected to be so because of the difficulty of the subject. Thus I made known these

important investigations to mathematicians, first in a short way and without putting forward any proof; but as it appeared afterwards that these questions were much in demand and that people did hardly believe them to be a new acquisition, I added their mathematical demonstration. At the same time, I was asked earnestly to publish the whole collection of my researches. To comply at least partly with this request, I have decided to publish the fundamental principles upon which my investigations are based. These new foundations of the theory of elliptic functions I now recommend to the favour of mathematicians.)

In *Arts.* 1—16 of the *Fundamenta*, the subject of rational transformation is dealt with on a purely algebraic basis. At first the problem of transformation is worked out in a detailed manner for the second, third and fifth degrees. In particular, with the help of a transformation of the second degree the general elliptic integral of the first kind is reduced to the normal form of Legendre. Further, for the transformations of the 3rd and 5th degrees the algebraic equations—later called Jacobi's modular equations—between the original and transformed moduli are given. Again it is shown that the multiplication of the elliptic integral can be reduced to two successive transformations.

With *Art.* 17 begins a systematic introduction of the inverse functions with the notations sin am u, cos am$u$, Δ am u. In *Arts* 18 *and* 19, by using the addition theorem of Euler and by proving that the

Jacobian elliptic functions, defined above, are expressible for the case of imaginary arguments in terms of the functions of real arguments and the complementary modulus, Jacobi establishes the double periodicity of his functions, gives their zeros, their infinities and the changes undergone by them with the increase of the argument by half periods. It may be mentioned that Jacobi gives 33 relations deduced from the addition theorem.

In *Arts.* 20—31, the subjects of transformation and multiplication are treated on the basis of the Jacobian functions. With the help of these functions, explicit transformation expressions as well as the values of the transformed modulus and the multiplicator are obtained and, from the development of special transformation formulae for divided periods, multiplication formulae are derived.

In *Art.* 32, Jacobi examines more clearly the modular equations, already given, for the transformations of the third and fifth degrees, and shows that they remain unaltered if arbitrary, but the same, transformations are carried out on the original as well as the derived modulus. An elegant expression for the multiplicator is obtained with the help of the differential equation of the 2nd order given by Legendre for the periods of the elliptic functions. Lastly, the remarkable theorem is given, viz. that the relation between the original modulus k and the derived modulus l admits of being replaced by a differential equation of the 3rd order which remains the same whatever the degree m of the transformation may be. In *Art.* 33 the most general solution of the aforesaid equation is given in the form

$$mKL+m'K'L'+m''KL'+m'''K'L=0,$$

where the m's are arbitrary constants and L and L$'$ are the same integrals for l and its complement l' as K and K' are for k and k'.

In *Arts.* 35–42 we have different expressions for the Jacobian elliptic functions and the modulus k. Jacobi introduces the elliptic function in the expression for the general algebraic transformation of degree n, and by proceeding to the limit $n=\infty$ he obtains expressions for his functions as quotients of infinite products. Further, expressions for k and the periods are obtained in terms of the quantity q which is $e^{-\pi\frac{K'}{K}}$. Then we have in *Art.* 41 a second type of representations for Jacobi's functions, viz. in Fourier series.

In *Art.* 42, the reciprocal functions $\dfrac{1}{\sin \operatorname{am}\left(\dfrac{2Kx}{\pi}\right)}$, etc. are expanded in Fourier series, and in *Arts.* 43–46, we have formulae for the powers of the Jacobian functions and of their reciprocals, and also a large number of infinite series in powers of q which are summed up.

Arts. 47–51 deal with the integrals of the second and third kinds. In *Art.* 47 Jacobi introduces, in the place of Legendre's integral of the second kind,

$$E(\phi) = \int_0^{\phi} \surd(1 - k^2 \sin^2 \phi)\, d\phi$$

$$= \int_0^{u} (1 - k^2 \sin^2 \mathrm{am}\, u)\, du,$$

the new function Z defined by

$$\frac{2K}{\pi} Z\left(\frac{2Kx}{\pi}\right) = \frac{2Kx}{\pi}\left(\frac{2K}{\pi} - \frac{2E}{\pi}\right)$$

$$-\left(\frac{2kK}{\pi}\right)^2 \int_0^{x} \sin^2 \mathrm{am}\left(\frac{2Kx}{\pi}\right) dx$$

so that $Z(u) = \dfrac{K.\, E(\phi) - E.\, F(\phi)}{K}$.

In the place of Legendre's integral of the third kind,

$$\Pi(n, c, \phi) = \int_0^{\phi} \frac{d\phi}{(1 + n \sin^2 \phi) \surd(1 - c^2 \sin^2 \phi)},$$

Jacobi introduces the function

$$\Pi(u, a) = \int_0^{u} \frac{k^2 \sin \mathrm{am}\, a \cos \mathrm{am}\, a\, \Delta\, \mathrm{am}\, a.\, \sin^2 \mathrm{am} u}{1 - k^2 \sin^2 \mathrm{am}\, a \sin^2 \mathrm{am}\, u}\, du$$

and proves the theorem

$$\Pi(u, a)-\Pi(a, u)=uZ(a)-aZ(u).$$

The fundamental properties of $Z(u)$ are investigated in *Arts.* 47 *and* 48; in *Art.* 49 the remarkable theorem is enunciated: that the elliptic integral of the 3rd kind of Legendre, which contains three variables, admits of being reduced to functions of only two variables. The proof is based on series, and, in terms of the function Θ, the theorem is

$$\Pi(u, a)=u\frac{\Theta'(a)}{\Theta(a)}+\frac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)},$$

In *Art.* 51 series for the 3rd integral are given.

The theta function Θ is introduced for the first time in *Art.* 52 and in the remaining part of the *Fundamenta* this function is predominant. The function Θ is introduced by the equation

$$\Theta(u)=\Theta(0).\,e^{\int_0^u Z(u)\,du}$$

and the integrals of the second and third kinds are expressed in terms of Θ by the relation

$$Z(u)=\frac{\Theta'(u)}{\Theta(u)},\quad \Pi(u, a)=u\frac{\Theta'(a)}{\Theta(a)}+\frac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)}.$$

Using these, the addition theorem of the integrals of the third kind for the argument as well as the parameter is proved.

For example, in *Art.* 55, we have

$$\Pi(u, a)+\Pi(v, a)-\Pi(u+v, a)$$

$$= \tfrac{1}{2} \log \frac{\Theta\,(u-a)\,\Theta\,(v-a)\,\Theta\,(u+v+a)}{\Theta\,(u+a)\,\Theta\,(v+a)\,\Theta\,(u+v-a)}$$

$$= \tfrac{1}{2} \log \mathrm{T}, \text{ where T is}$$

$$\frac{1-k^2 \sin \operatorname{am} a \sin \operatorname{am} u \sin \operatorname{am} v \sin \operatorname{am} (u+v-a}{1+k^2 \sin \operatorname{am} a \sin \operatorname{am} u \sin \operatorname{am} v \sin \operatorname{am} (u+v+a)}.$$

and

$$\Pi\,(u,a)+\Pi\,(u, b)-\Pi\,(u,a+b)$$

$$= \tfrac{1}{2} \log \mathrm{W}+uk^2 \sin \operatorname{am} a \sin \operatorname{am} b \sin \operatorname{am} (a+b), \text{ W being}$$

$$\frac{1-k^2 \sin \operatorname{am} u \sin \operatorname{am} a \sin \operatorname{am} b \sin \operatorname{am} (a+b-u)}{1+k^2 \sin \operatorname{am} u \sin \operatorname{am} a \sin \operatorname{am} b \sin \operatorname{am} (a+b+u)}.$$

In *Art.* 56 the theta function with an imaginary argument is first considered and it is shown how the function is reducible to one with real argument, the quantity q being different for the two functions. Further, Jacobi gives, with the help of definite relations for the linear transformation of Θ, the reduction of an arbitrary integral of the 3rd kind to such a one of the same kind as has its argument and parameter both real and both lying between 0 and -1.

In *Art.* 61, the new transcendental H is introduced by the equation

$$\frac{\mathrm{H}\left(\frac{2\mathrm{K}x}{\pi}\right)}{\Theta\,(0)} = \frac{2\,q^{\frac{1}{4}} \sin x \prod\limits_{n=1}^{\infty} (1-2q^{2n} \cos 2x+q^{4n})}{\prod\limits_{1}^{\infty} (1-q^{2n-1})^2}.$$

It is shown that

$$\sin \operatorname{am} u = \frac{1}{\sqrt{(k)}}\,\frac{H(u)}{\Theta(u)},$$

$$\cos \operatorname{am} u = \sqrt{\left(\frac{k'}{k}\right)}.\,\frac{H(u+K)}{\Theta(u)},$$

$$\Delta \operatorname{am} u = \sqrt{(k')}.\,\frac{\Theta(u+K)}{\Theta(u)},$$

$$\sqrt{(k)} = \frac{H(K)}{\Theta(K)},\quad \sqrt{(k')} = \frac{\Theta(0)}{\Theta(K)}.$$

The fundamental properties of $H(u)$ and $\Theta(u)$ are given and it is shown how by increasing the argument by half a period one is simply expressible in terms of the other. From Art. 62 onwards, we have transformations of infinite products into series, the results,

$$\Theta\left(\frac{2Kx}{\pi}\right) = 1 + 2\sum_{1}^{\infty} (-1)^n q^{n^2} \cos 2nx,$$

$$H\left(\frac{2Kx}{\pi}\right) = 2\sum_{1}^{\infty} (-1)^{n-1} q^{(n-\frac{1}{2})^2} \sin (2n-1)x,$$

and the famous series for the period from the relations

$$\Theta(0) = \sqrt{\left(\frac{2k'K}{\pi}\right)},\ \Theta(K) = \sqrt{\left(\frac{2K}{\pi}\right)},\ H(K) = \sqrt{\left(\frac{2kK}{\pi}\right)}$$

$$\frac{2K}{\pi}\,H'(0) = \frac{2K}{\pi}\,\sqrt{\left(\frac{2kk'K}{\pi}\right)}.$$

The different analytical forms of the elliptic functions give a series of interesting identities between infinite products and infinite series in powers of q. The book concludes with Jacobi's proof, on the basis of the above identities, of Fermat's theorem that every integer admits of being expressed as the sum of four squares.

The following details about the genesis and publication of the *Fundamenta* will be of interest:—

(1) Jacobi seems to have made up his mind as early as November, 1827 to bring out his researches on elliptic functions in the form of a book which later he wanted to publish in two parts; the first part being the *Fundamenta*.

(2) Writing to Legendre on the 9th September, 1827, Jacobi says: "My researches shall be collected in a small work of about 200 pages in 4° which shall be printed and of which the printing is to be commenced immediately. It shall have for title: *Fundamenta Nova Theoriae Functionum Ellipticarum*. May be, I shall be sufficiently fortunate to present it to you.

"It is necessary to admit, Sir, that I am a little tired of the matter which has occupied me nearly day and night during 18 months".

(3) Writing to Legendre on 14-3-'29, Jacobi says: "The printing of my work is delayed, because the press is at a distance of 200 leagues from Koenigsberg; but for that the work would have been in your hands; however, I hope to be able to send it to you in very little time".

(4) In the letter to Legendre of 23-5-'29, we read: "The printing of that (my work) has been finished......You will oblige me much, Sir, if you will have the kindness to hand over to Messrs. Poisson, Fourier and Cauchy the copies of my work which will remain after that which I present to you".

(5) The preface of the *Fundamenta* is dated February, 1829; the book appeared in April after the death of Abel on the 6th of that month; Jacobi presented a copy to the Prussian Ministry of Education on the 18th May and a copy to the Academy of Sciences of Paris in June; the Academy asked Poisson to submit a report which was read at the sitting of the 21st December, 1829.

At the same time that Jacobi became a Ph. D. he was appointed to be a lecturer at the University of Berlin and actually delivered his first lecture on the theory of curved surfaces and curves in space. The impression produced by Jacobi of his talents as a teacher was so great that only after half a year in Easter, 1825 the Ministry of Education granted his application for appointment at Koenigsberg University as lecturer in the place of the newly deceased Professor of Mathematics, Wrede. The appointment offered to Jacobi a great scope for his ability and the vacant chair of Mathematics meant greater chance of promotion than at Berlin. In 1827, Gauss so highly esteemed Jacobi on account of his researches on cubic residues that he was being talked of in high Government circles as fit for promotion to an Assistant Professorship which went to him in spite of much opposition for reasons which will be clear from the following quotation: "So stood Jacobi in very short time by the side of Gauss as one of the leaders for the building up of mathematical Science; however, he still remained a privat-docent with the salary of 200 thalers a year, although he was the only teacher of Mathematics at Koenigsberg. But when the Prussian Minister asked for the opinion of the Faculty regarding the appointment of Jacobi as an Assistant Professor, the Faculty expressed no doubt about his distinguished scientific achievements; however, notwithstanding this, the Faculty expressed itself as against his promotion" in the face of an older man like F. Neumann or Dove.

The Curator of the University, however, declared the misgivings of the Faculty as improper, and the favourable opinion which he expressed about Jacobi's scientific achievements as well as the enthusiastic report by Legendre to the Academy of Paris on Jacobi's works determined the government to give the appointment to Jacobi and he became Assistant Professor in December, 1827. Franz Neumann felt humiliated at the time but his regard and admiration for Jacobi continued up to the end of his life. With the publication of the *Fundamenta Nova*, Jacobi was recognized to be indisputably as one of the German mathematicians, second only to Gauss. But the incessant labour of years had exhausted Jacobi both physically and mentally and he applied for leave for the whole of the summer semester which was granted. After spending some months with his friends and relatives at Berlin he travelled in Thuringia and thence went to Paris where he stayed from the end of August up to the middle of October "in full enjoyment of Nature and Art, and at the same time in constant scientific touch with Legendre, Fourier, Poisson and other eminent mathematicians and physicists some of whom later survived him".

Legendre had always had a real attachment, to Jacobi since the time when Jacobi wrote to him on the 5th Aug., 1827, his first letter. A long correspondence followed and its perusal in the pages of *Crelle's Journal*, Bd. 80, throws much light on the difficult question of the comparative claims of Abel and Jacobi as the first discoverer of the theory of elliptic functions. Soon after the publication of the *Fundamenta Nova* Poisson submitted a long report on it to the Institute of France with the result that, although no prize had been announced for elliptic functions, the grand prize for Mathematics was divided in 1830 equally between Jacobi and the heirs of Abel. At about this time Jacobi was made an ordinary Professor. As regards the claims mentioned above, we can not do better than give the opinions of three or four great mathematicians and will begin with the following remarks due to Briot and Bouquet found in the preface to their *Théorie des fonctions elliptiques*, 1875:—"Abel as the first, in 1826, has considered elliptic functions, properly so called, which are the inverses of these (elliptic) integrals, and has recognized the existence of two periods. Towards the same time, Jacobi was occupied with the same subject and the immortal works of these two great mathematicians have appeared in the first volumes of the *Crelle's Journal*. The researches of Abel refer not only to elliptic transcendentals but also to other transcendentals of higher order; he has discovered on this subject a theorem which one regards as one of the most beautiful conquests of modern Analysis.....It is by following the path opened by Abel that a large number of eminent mathematicians of our times have enriched the science with their brilliant discoveries."

Next let us see what an eminent countryman of Abel, viz. C.—A. Bjerknes* has said. "In the sad circumstances in the midst of which Abel found himself on his return to his country, it was only in September, 1827, that there appeared in the *Crelle's Journal* the first part of his memoir cited earlier in this book and which is celebrated now as the *Récherches* This memoir, on which he worked during his stay at Paris, and to which he makes clear references in his letters to Holmboë and Crelle, was sent to the editor on his return to Berlin; although it had not been completed and admitted of a considerable extension, the new theory of elliptic functions was found established by means of this epoch-making work.

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* Bjerknes; *Niels—Henrik Abel*, Paris, 1885, pp. 124–126.

"In the same month, there appeared in the *Astronomische Nachrichten* of Schumacher the enunciations of Jacobi concerning the theory of transformations; the first was dated the 13th June and the second, the 2nd August. The whole did not occupy in the journal more than a sheet, but, in spite of the absence of developments, it was none the less a work of great importance, One finds there a generalization of the old theory of transformations such as had existed since Legendre. However, the first generalization which was published in the paper of June......had been made by Legendre himself without Jacobi having had knowledge of it. The paper of August, which, among other things, contains as supplement a communication made to Legendre on the 5th of that month, is on the contrary of a very high value. Jacobi had discovered here a truth of a great profundity relatively.

"In spite of this, as we have said above, the discovery of Jacobi, such as it presented itself at this time, cannot be considered to be other than a very large complement of the old theory; one ought to envisage it as an extension of the doctrine of Legendre. For, as a resort to an attentive study of the articles of Jacobi and of his letter to Legendre will show, neither of the two great and capital ideas, the idea of inversion and that of the existence of double period city, appeared to have entered his field of vision at this time. Of course, one does not perceive any indication whatever of the ganeralization which results for elementary simply periodic functions, and their extension into certain *new* functions endowed with a kind of higher periodicity.

"If one comes to designate the date of September as the remarkable epoch at which from the fact of the two publications the theory of elliptic functions was simultaneously discovered by Abel and Jacobi, one adopts in this case a wrong tradition. For all that relates to Abel, this discovery existed already, complete not only in principle but also its most important developments. By the ideas which he had made known, by his plan presented with clarity and with order, by the indications which he had given and by the choice of his notations—and, added to all these, by the thoughts and methods which he had made known beforehand in his memoir of Freiberg—Abel dominated, we may say, the new doctrine on all its fundamental points and in all its extent. The second part of the *Recherches* promised for the next issue would not be considered necessary for the completion of the theory after the establishment of the principles and the the luminous indications regarding what was to follow ; for, without doubt the mathematicians of much inferior rank could on these foundations continue the edifice".

Bjerknes emphasizes certain facts in support of his opinion. The two papers referred to above were followed by a paper, dated the 18th November, in which Jacobi gave the demonstrations of the two theorems. After the date, viz. 20th September, on which the part of *Crelle's Journal* containing the first part of Abel's *Recherches* came out, nearly two months was taken by Jacobi to send to Schumacher his proofs—and this delay was in spite of the repeated requests of Schumacher for them. Two other points are mentioned by Bjerknes. He thinks it rather strange that Jacobi should have sent a purely mathematical contribution to a small Astronomical Journal and not to *Crelle's Journal*, of which he was a collaborator. Bjerknes seems to imply that Crelle would have rejected the enunciations without proofs accompanying them. The second point is made in the following remarks: "In his memoir of the 18th November, where he introduces his notation of inversion, Jacobi cites nowhere the name of Abel. This abstention from citing the origin when the matter relates to an object of such high importance, or at best if the ideas can be said to be of the same time—this procedure does not explain in a precise manner the independence of his discovery and his right of coproperty—this behavior cannot, in fact, be explained away only as a fault of a young mathematician". The opinion of Bjerknes, putting it in blunt language, is that Jacobi guessed out the theorems by a process of induction and he seems to imply that the ideas of Abel were used by Jacobi to formulate his proofs.

In connection with the rival claims of Abel and Jacobi, three issues arise :—

(1) Who was the *discoverer* of the elliptic functions?

(2) Who *introduced* the elliptic functions into Analysis ?

(3) What was Jacobi's share in the *development* of the theory of elliptic functions?

We proceed to answer the above questions, one after the other, briefly indicating the reasons for each answer.

The answer to the *first* question is: *Gauss*; for reasons * see pp. 50 and 51 of this volume. The answer to the *second* question is: *Abel*. The chief reason for this answer is that *almost all* that has been quoted above from Bjerknes's book is universally accepted by mathematicians, † and there is no doubt in our mind that Jacobi, who began his investigations on elliptic integrals in the winter of 1826—1827, was *not* in possession of the fundamental properties of elliptic functions *before* the 20th Sepember, 1827, when the first part of Abel's *Recherches* was published. We do not answer definitely the question, whether Jacobi obtained his results of June and August independently of Abel; it will be wise to answer this question in favour of Jacobi. But Jacobi's claim * to have introduced, along with Abel, elliptic functions into Analysis is not justified, strictly speaking.

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* See also Koenigsberger's *Festschrift*, pp. 537—538; also Gauss's *Werke*, Bd. X_1, pp. 251—286. The identity between a certain series and an infinite product, which Jacobi claimed to be the most important and most fruitful of all his own discoveries, was known to Gauss as early as 1801.

† See p. 146 of this volume; also Koenigsberger's book, pp. 36 and 37. All the publications of Jacobi on the properties of *elliptic functions* were *after* November, 1827. Abel was the man whose works influenced most Weierstrass's researches. "Read Abel" was Weierstrass's first and last advice to every one. (See Mittag-Leffler's paper on Weierstrass in *Acta Math*, Vol. 21, p. 81).

As regards the *third* question, there is no doubt that after the elliptic functions had been discovered by Gauss and introduced into Analysis by Abel, Jacobi did more than any one else to develop the theory; the nearest to him in this matter comes Weierstrass. The various applications, by Jacobi, of elliptic functions to the theory of Numbers, Geometry and Mechanics made those functions appear important in the eyes of the French school of Mathematics—the dominant school of his time.

After 1830, Jacobi became famous as the builder of the first school of Mathematics in Germany which he conducted with great success for about 17 years. Amongst his pupils may be counted Richelot, Rosenhain, Göpel, Hesse, Weierstrass, and even foreigners like Hermite and Cayley. Jacobi was much attached to his pupils and after getting acquainted with any young and brilliant mathematician Jacobi never lost sight of him in his future career.

The output of research due to Jacobi's own labours was huge; almost every branch of Mathematics received important contributions from him towards the extension of its bounds. A list of his principal memoirs, in addition to those already enumerated on pp. 169–192 is given at the end of this chapter. It will be seen that the memoirs of Jacobi relate to partial fractions; approximate quadrature; various parts of the theory of numbers and the applications of elliptic functions to that theory; determinants, including what are now called functional determinants which were discovered by Jacobi; geometry of surfaces of the second degree; differential geometry, specially the theory of geodesic lines; ordinary differential equations, total and partial differential equations; transformation of multiple integrals; attractions of ellipsoids; figures of equilibrium; Poncelet's polygons; higher plane curves and the applications of elliptic functions to them; analytical dynamics; calculus of variations; transformation of homogeneous functions; elimination; the simultaneous values which a number of algebraic equations satisfy; the inversion of series; spherical harmonics and allied functions; celestial mechanics; Astronomy.

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* From a paper, "Zur Geschichte der elliptischen und Abelschen Transcendenten," (*Werke*, Bd. 2, pp. 514–521) which was written in 1847, it is seen that Jacobi speaks of "die elliptische Funktion $x =$ sin am u, welche Abel und ich in die Analysis eingeführt haben," (See p. 516).

The industry of Jacobi had become proverbial in the world of Mathematics. His hard work, continued for years, brought about a break-down in 1839: he suffered from headache and his nervous condition made work impossible. He was, therefore, constrained to take leave in summer in order to go to a bath for cure. He left Königsberg in March, went to Marienbad for a stay of several weeks and, after visiting Schweins, the teacher of Steiner, at Heidelberg and Gauss at Göttingen on his return journey, reached Königsberg after 7 months' absence. Jacobi continued his research activities for a few years more but family troubles produced a change. His father had died in 1832. And in about 1840 very great pecuniary losses resulted in the dissolution of his family business and he lost not only his whole fortune but also his mother lost hers. Bessel informs Gauss of these calamities and says: "Luckily such a talent cannot be ruined, but I would have liked him to have the feeling of freedom which the possession of wealth guarantees." As a matter of fact, according to Dirichlet, Jacobi did not show his loss by any change in his demeanour. "Any one who saw him at that time could not perceive the slightest alteration in his temper; he spoke with the same interest as before of scientific matters and complained only of the interruption that his unexpected journey would cause in the investigations with which he had been very busy."

The journey, referred to above, was to Manchester and Paris in 1842. As desired by the King, Jacobi and Bessel visited the British Association at Manchester in July, 1842, and Jacobi communicated in his address before the Association his latest researches about the principle of the last multiplicator. On his return, Jacobi gave his great course of winter-lectures of 1842—1843 on the integration of differential equations. He went on with his researches on the problem of three bodies and Abel's theorem and communicated to the Berlin Academy an investigation on new developments in the calculation of perturbations. But at this time his health completely broke down. When the King heard of this he wrote to Jacobi: "With lively regret have I come to know of your ill health but at the same time I am set at ease by the assurance that you will be restored to your normal condition after stay in a milder climate", and granted to Jacobi all that Jacobi's friends had wanted. After 5 months' residence in Rome and Naples in the company of his pupil, the young Dr. Borchardt, and Dirichlet, Jacobi returned to Berlin in the end of June, 1844. A cabinet order, dated the 20th August, 1844, permitted him to stay on at Berlin until his health was completely restored; he was nominally attached to the University but was to give only such lectures as his health and research activities would allow; and further he was given a special allowance of 1000 thalers a year from the Privy Purse.

Jacobi's stay in Berlin was marred by certain unfortunate incidents of a political character. When the political unrest of March, 1848 arose, Jacobi, who had no regular post at the University, became afraid of losing, because of a political revolution, that considerable part of his income which he received exclusively because of the King's favour; and he requested the Minister of culture to make him an ordinary professor and thus remove him from his special position. This request was refused on the ground that the Faculty was opposed to having more than three ordinary Professors and also felt that Jacobi's appointment would not have a healthy influence on the work of the reorganization of mathematical teaching.

Jacobi began to take part in politics, partly because of the advice of his medical friends who thought that political activity would have a beneficial effect on his nervous system. He was introduced by Dove on the 21st April, 1848, to a club of Liberal moderates, called the constitutional club, and offered himself for selection by the club as a candidate for the parliamentary elections which were be held in Berlin on the approaching first of May. Jacobi delivered an address before the club and it was such a great success that his rivals in the club combined to discredit him. Three charges were levelled againt him, viz. (1) that he had been in the past servile and that at the time he had made a sudden somersault, (2) that he had been from before a fullfledged Jacobin and (3) that he had been always politically indifferent. All sorts of incidents from his past were mentioned to substantiate one charge or another. Some said that he had always been specially in the good books of the King and that he had always attached himself to those in power. Jacobi delivered before the club on the 25th April a powerful speech in his defence and on the 27th the proposal to remove his name from the list of candidates was withdrawn. The debates about Jacobi produced much sensation in Berlin for about a week, but Jacobi failed to get into Parliament.

The result of Jacobi's political activities on his academic position was unfavourable. In May, 1849, a year after Jacobi's political activities had ceased, he received a letter from the Minister Ladenberg asking whether his health was sufficiently improved to admit of his returning to Köenigsberg, and soon after that his special allowance was stopped. Jacobi was thus compelled to remove his wife and 7 small children to Gotha to live there with his friend Hansen and to remove himself to a room in the London Hotel of Berlin. At the end of 1849, he received a brilliant offer from the Vienna University, but by the intercession of v. Humboldt the King was persuaded not to let Jacobi leave Prussian service, and his special allowance was restored with retrospective effect.

After returning from a Christmas visit to his family in Gotha, Jacobi became ill of grippe from which he appeared to recover quickly. But on the 11th February 1851, he had a relapse*. Four days before his death he complained to his friend Dirichlet about the mishaps which had delayed the publication of many of his extensive researches: I see clearly that "I should not delay any longer to give to the public those works, to which I devoted so great a part of my best power, in time for them to be grasped into the current of science." On the 18th February, 1851, at 11 A. M. he died peacefully, and was buried at the Trinity Chirch before the Halle Gate in Berlin.

List of the principal papers of Jacobi in addition to the 21 already mentioned on pp. 169—192. :—

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* According to Dirichlet this last illness was small-pox.

I. "Theorie der elliptischen Functionen aus den Eigenschaften der Thetareihen abgeleitet" (nach einer Vorlesung Jacobis in dessen Auftrag ausgearbeitet von Borchardt in 1838; *Werke*, Bd. 1, pp. 499–538; also Ostwald's *Klassikern*).

II. "Ueber die Figur des Gleichgewichts" (*Werke*, Bd. 2, pp. 17–22).

III. "De functionibus duarum variabilium quadrupliciter periodicis, quibus theoria transcendentium Abelianarum innititur" (*Werke*, Bd. 2, pp. 23–50; also in Ostwald's *Klassikern*, as No. 64).

IV. "Note von der geodätischen Linie auf einem Ellipsoid und den verschiedenen Anwendungen einer merkwuerdigen analytischen Substitution" (*Werke*, Bd. 2, pp. 57–63).

V. "Ueber eine particuläre Lösung der partiellen Differentialgleichung

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} = 0\text{''}$$

(*Werke*, Bd. 2, pp. 191–216).

VI. "Ueber unendliche Reihen, deren Exponenten zugleich in zwei verschiedenen quadratischen Formen enthalten sind" (*Werke*, Bd. 2, pp. 217–288).

VII. "Sur la rotation d'un corps" (*Werke*, Bd. 2, pp. 289–352).

VIII. "Ueber die Substitution

$$(ax^2+2bx+c)\,y^2+2\,(a'x^2+2b'x+c')\,y$$
$$+a''x^2+2b''x+c''=0$$

und über die Reduction der Abelschen Integrale erster Ordnung in die Normalform" (*Werke*, Bd. 2, pp. 363–379).

IX. "Fragments sur la rotation d'un corps" [*Werke*, Bd. 2, pp. 425—512. These are from the MSS. left by Jacobi and have been published as the following papers. (1) Seconde mémoire sur la rotation d'un corps non soumis á des forces accélèratrices." (2) "Nouvelle théorie de la rotation d'un corps de revolution grave suspendu en un point quelconque de son axes" and (3) "Sur la rotation d'un corps de révolution grave autour d'un point quelconque de son axes."]

X—XI. Two papers in Latin on the transformation of double integrals, (*Werke*, Bd. 3, pp. 91—158 and pp. 159—189).

XII. A paper in Latin containing various theorems on the transformation and determination of multiple integrals (*Werke*, Bd, 3, pp. 191—268).

XIII. "De eliminatione variabilis e duabus aequationibus algebraicis" (*Werke*, Bd. 3, pp. 295—320).

XIV. "De formatione et proprietatibus determinantium" (*Werke*, Bd. 3, pp 355—392).

XV. "De determinantibus functionibus" (*Werke*, Bd. 3, pp. 393—438).

XVI. "Zur Theorie der Variationsrechnung und der Differentialgleichungen" (*Werke*, Bd. 4, pp. 39–55).

XVII. "Ueber die Reduction der Integration der partiellen Differentialgleichungen erster Ordnung

zwischen irgend einer Zahl von Variabeln auf die Integration eines enizigen Systems gewöhnlicher Differentialgleichungen". (*Werke*, Bd. 4, pp. 147–255).

XVIII. "Dilucidationes de aequationum differentialium vulgarium systematis earumque connexione cum aequationibus differentialibus partialibus linearibus primi ordinis" (*Werke*, Bd. 4, pp. 147—255).

XIX. "Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi" (*Werke*, Bd. 4, pp. 317—509).

XX. "Nova methodus aequationes differentiales partiales primi ordinis inter numerum variabilium quemcunque propositas integrandi" (*Wetke*, Bd. 5, pp. 1—189).

XXI. "Ueber diejenigen Probleme der Mechanik in welchen eine Kräftefunction existirt und ueber die Theorie der Störungen" (*Werke*, Bd 5, pp. 217—395).

XXII. "Ueber die vollständigen Lösungen einer partiellen Differentialgleichung erster Ordnung" (*Werke*, Bd. 5, pp. 397—438).

XXIII. "Ueber die Integration der partiellen Differentialgleichungen erster Ordnung zwischen vier Variabeln" (*Werke*, Bd. 5, pp. 439—464).

XXIV. "Ueber die Entwicklung des Ausdrucks $[a^2-2a a' \{\cos \omega \cos \phi+\omega \sin \phi \cos (\theta-\theta')\}+a'^2]^{-\frac{1}{2}}$" (*Werke*, Bd. 6, pp. 148—155).

XXV. Untersuchungen über die Differentialgleichungen der hypergeometrischen Reihe" (*Werke*, Bd. 6, pp. 184—202).

XXVI. "De compositione numerorum et quatour quadrats" (*Werke*, Bd. 6, pp. 245—251).

XXVII "Uber die Kreistheilung und ihre Anwendung auf die Zahlentheorie" (*Werke*, Bd. 6, pp. 254—274).

XXVIII. "Ueber die complexen Primzahlen welche in der Theorie der Reste der 5 ten, 8 ten und 12 ten Potenzen zu betrachten sind". (*Werke*, Bd. 6, pp. 275—280)

XXIX. "Ueber die Zusammensetzung der Zahlen aus ganzen positiven Cuben; nebst einer Tabelle für die kleinste Cubenzahl, aus welcher jede Zahl bis 12000 zusammengesetzt werden kann." (*Werke*, Bd. 6, pp. 322 – 354).

XXX. "Ueber ein leichtes Verfahren, die in der Theorie der Säcularstörungen vorkommenden Gleichungen numerisch aufzulösen" (*Werke*, Bd. 7, pp. 97 – 144).

XXXI. "Versuch einer Berechnung der grossen Ungleichheit des Saturns nach einer strengen Entwickelung" (*Werke*, Bd. 7, pp. 145).

XXXII. "Untersuchungen über die Convergenz der Reihe, durch welche das Keplersche Problem gelöst wird" von Franz Carlini, bearbeitet von Jacobi 1850. (*Werke*, Bd. 6, pp. 189 – 245).

CHAPTER V.

WEIERSTRASS

Karl Theodor Wilhelm Weierstrass was born on the 31st October, 1815, in Ostenfelde, District Warendorf, Province Münster, as the eldest son of Wilhelm Weierstrass (1790—1869) and his wife of the family of von der Forst. After him were born his brother, Peter (1822—1904), who became later a teacher in the Pro-gymnasium or middle school at Krone, and two sisters, Klara (1823—1896) and Elise (1826—1898). Weierstrass's mother died in 1826 and his father married again in 1827. The step-mother died in 1859 and Weierstrass's father spent the last ten years of his life in the house of Weierstrass at Berlin.

The father of Weierstrass was a well-educated man with versatility of knowledge; he knew both Physics and Chemistry and spoke and wrote French as if it were his mother tongue. At the time of Karl's birth, Wilhelm Weierstrass was tax-collector in the service of the French and before that he had been a teacher since 1808. A short time after the birth of Karl, his father went with his family to Westernkotten in Westphalia, where he had got a post in the Salt Works. He was a Catholic, having been converted from

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〈 1815—1896 〉

Protestanism, and so were all his children also Catholics, but the family was not bigotted. It may be noted here that neither Weierstrass nor his brother nor any of his sisters ever married. This was probably because of their economic condition before 1864.

As there was no school at Westernkotten, Weierstrass received his earliest education at the school at Münster which was close by. At the age of 14 years, he was admitted into the 6th class of the Catholic Gymnasium or secondary school at Paderborn where the usual course of studies covered 6½ years. But Karl was promoted immediately from the 4th to the 2nd class, thus shortening his school period by a year. He left the Gymnasium in August, 1834, with brilliant certificates. During his stay at the school, Karl had received several times as many as six prizes and once he got seven prizes. He always got the first prize in his mother tongue, German, and in two other subjects which changed from time to time in the list of the three subjects, Latin, Greek and Mathematics. He also got prizes in other subjects with the exception of hand-writing in which he never got a prize. By an irony of fate, later on in his career, he was at one time teacher of hand-writing. As early as in his 15th year, Weierstrass had used his mathematical ability to earn some thing by keeping the accounts of a rich woman who dealt profitably in ham and butter. He was so precocious that even in his second class he studied closely Integral Calculus.

The distinguished success of Karl at the Gymnasium fired his ambition and raised high expectations of his future in the minds of all others. His father chose for him his career and sent him to the Bonn University to carry on his financial and juristic studies there so that he might later find a place in the higher ranks of the Prussian bureaucracy. Weierstrass's life at the University, where he stayed four years from 1834 to 1838, was a rather wild one; although he did not altogether give up Mathematics. His youthful adventures, his reputation for drinking and unconquerable fencing, appear to us strange, in view of what he became later, one of the greatest mathematicians of the 19th century. And he attended very few lectures, financial, juristic or mathematical. Only during one semester, he attended a course of lectures delivered by the famous Plücker who had been since the year 1836 Professor of Mathematics and Physics at Bonn. Before Plücker came, the post of Professor of Astronomy, Mathematics and Physics was held by Karl Dieterich von Münchow, who had shown a liking for Weierstrass and had advised him from time to time about his study of Mathematics; but Weierstrass had never attended any lectures by Münchow.

Suddenly one day in the late autumn of 1838, Weierstrass returned to the house of his father, who felt that all his expectations of Karl had crashed to the ground. Karl appeared to be in very poor health of body and mind, although the exact cause of his wretched condition was never known. The family of Karl thought that 4 years of his life had been wasted and began to form plans for his future. Karl remained with his father for about six months, and then with the advice of the chief judicial officer of Paderborn, Schlechdenthal, who had been much interested in Weierstrass during his student days at the Paderborn Gymnasium, it was decided that Weierstrass should leave the University and seek admission into the Münster Academy with a view to qualify himself for the State Examination for the diploma in teaching, so that after the examination he might become a school teacher though not a Ph. D. So, on the 22nd May, 1839, Weierstrass entered the Münster Academy where he attended only the lectures of Gudermann (1798—1852) who was the teacher of Mathematics there. Gudermann was ever afterwards remembered by Weierstrass with the deepest reverence and gratitude.

On the 29th February, 1840, Weierstrass applied to the president of the Examination Commission for permission to appear as a candidate. and stated frankly and fully his past experiences at the Bonn University. The permission was granted for appearance at the examination which was to be partly oral and partly written and was to be very searching and comprehensive; and on the 2nd May, 1840, Weierstrass recieved the following order :—

"On your application for admission to the exmination pro facultate docendi having been granted by the Royal Ministerial-Commission at Berlin, the questions for the written examination will be as indicated below :-

1. (Philological task in Latin)

Explicitur, qui viri rempublicam Romenorum liberam domi militiaeque omnium maxime adiuverint atque auxerint.

2. (Mathematical task in German or French)

According to the problems on the attached sheet to be reattached to the answer-book.

3 (Pädagogical task):

Difference between the Utility principle and Humanity Principle in education and instruction, as the one or the other can be applied to every subject of teaching (with particular reference to mathematical teaching).

The answers—in folio on loose sheets—are to be sent within six months. In the case of each answer, a certificate is to be added at the end in the form 'that the answer has been prepared by yourself alone and without any help from any one'. Also are all literary auxiliaries to be completely and conscientiously stated. After the despatch of the answers, the dates of the oral examination and of the test—lectures will be determined". The mathematical part consisted in the solution of the following three problems which were chosen by Gudermann, one of the examiners :—

1. The modular functions (elliptic functions) can be looked upon as fractions, and the denominators as well as the numerators can be represented as products of infinite number of factors and also as series in different ways, but always with the help of the method of limits. Since many objections have been raised against this method, therefore a representation of the

quantities in the forms mentioned, and based on differental equations obtained before—hand, is desired. Besides, is desired an expansion of the same in series of powers of the argument, and in this case is to be given in addition to the recurrence formulae also the independent determination of the unknown coefficients up to a certain extent. Also, the known differential equation for the transformation of the modular functions is to be deduced.

Note. This problem, which in general is far too difficult for a young analyst, has been set to the candidate at his express request and with the consent of the Commission.

2. From Elementary Geometry: In a triangle should a second triangle be so described that the vertices of the second lie on the sides of the first and the angles of the second are bisected by the straight lines which join the opposite vertices of the triangles.

3. From Higher Mechanics: If an attractive force works as a central force so that it varies as the inverse square of the distance from the centre, then is desired the equation between time and space as also the form of the orbit.

The candidate has previously studied with the permission of the Commission a theme desired by him for being set".

After the end of the written examination, in April of the next year the oral examination was held on the 23rd from 8 A. M. to 10 A. M., followed by two test lectures to the first class of the Gymnasium from 10 A. M. to 1 P. M. and from 3 P M. to 5 P. M. There was again an oral examination from 3 P. M. to 6 P. M. on the 24th. Weierstrass received the diploma and the following additional certificate:— "The first answer relating to a theme, desired by the candidate himself, displays a distinguished talent for the problems of higher Analysis. With the solution of the problem, he has broken an entirely new path in the theory of modular functions and has, in this manner, as was to be expected, arrived at the series and factor-developments known up to now, and also at entirely new results. The solution of the two other problems is also fully satisfactory and shows likewise the mathematical ingenuity of the author".

It is related that Gudermann had put down in the certificate the words: "the candidate enters consequently in his own right into the category of famous discoverers"; but they were scored out as not suitable for an official document. Mittag-Leffler says: No mathematician, who has studied Weierstrass's first answer, can reject the enthusiastic opinion of Gudermann.

In the autumn of 1842, a new post of Teacher of Mathematics and Physics was created at the Progymnasium of Deutsch Crone in West Prussia and Weierstrass was sent there as Assistant Teacher but soon became Teacher. In addition to Mathematics and German, Geography, Hand—writing and later on Gymnastics were also the subjects in which he gave instruction.

In the autumn of 1848, he was appointed Teacher at the Gymnasium of Braunsberg in East Prussia, where he remained until 1855.

Of the first 40 years of his life, Weierstrass writes as follows in a letter* to P. Du Bois-Reymond in 1875, after he had become great and famous:—"When I resolved, after seven University-semesters, gaily and happily spent, to occupy myself from that time with Mathematics as my business in life—which made my friends doubt my sense of responsibility and prepared sorrow for my father, who had placed before me the aim to become a Regierungsrat at the latest within ten years of the expiry of my Triennium—I was fated to do penance for that resolution first with 14 years of exile in the land of Velaten and Obotriten. That was a bad time, whose infinite emptiness and tedium, would have become intolerable without the hard work which that time brought with it, which, however, was bound to make a man, who had until then led a life full of glorious days, a recluse, which I really became and remained, even if I was taken to be a rather good fellow by the circle of landlords, advocates and young officers surrounding me; in those days—which exactly covered those years of my life—which are otherwise called the most beautiful—I lost the pleasure of written communications—the present did not offer anything worth speaking of and it was not my way to speak of the future. In addition to this, there was another thing which to the present generation appears hardly credible, viz. in those days the recreation of a regular correspondence was a luxury which a school-master with 29 thalers a month for working 30 hours a week could not afford ; in the beginning of 1840 a letter from Koenigsberg to Cologne cost 17 silvergroschen. Excuse me, dear colleague, that I entertain you with this tale of misery; but the memory of those 14 years s exactly in these years become more lively".

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* See *Acta Math.*, Vol. 39, pp. 209-210.

Klein points out that during the first forty years of his life Weierstrass was intimately associated with Catholics; the neighbourhood in which he spent his childhood and early youth was populated mainly by Catholics; all the schools where he served as school-master or studied as student were Catholic. Another point emphasized by Klein is the obscurity of the places where Weierstrass served as school-master. They were "remote from all scientific life, almost inaccessible to any mathematical stimulus", they were small, nay, smallest places of which, otherwise, one would not have heard the names. As Mittag-Leffler says sarcastically, it is in one of these places that the man, who had been declared by Gudermann to be the equal of famous discoverers, spent six years in teaching the elements of Mathematics to little boys.

Of Weierstrass's hard work as a researcher the following story is related of his Braunsberg days : "One winter-morning at about 8 the Director of the Gymnasium, Schultz, became aware that the teacher of the

Prima class had not appeared for his teaching work. As Weierstrass had to take the class in that hour and lived on the premises, the Director himself went to his rooms to see why he was hindered from appearing. There he saw Weierstrass quite active in the work which he had begun on the preceding evening: he had continued to work throughout the whole night and did not realize that the night was over and that it was the morning of the next day. On the remark of the Director that it was time for him to begin his class-work, Weierstrass begged him not to mind if he did not observe his school hours punctually: he had attacked a difficult problem and hoped soon to be able to surprise the world with an important discovery.*"

We proceed now to describe as briefly as possible the first ten papers of Weierstrass, ending with the paper, "Theorie der Abel'schen Funktionen" which raised him at once out of his obscurity :—

(I)

"Ueber die Entwicklung der Modular—Funktionen" (Part published in *Crelle's Journal*, Bd. 52, pp, 346-379; fully published for the first time in *Werke*, Bd. 1, pp. 1—49; the paper was written at Westernkotten in summer, 1840, and presented in the autumn of the same year to the Examinations—Commission at Muenster).

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* See Killing's Rektorats—Rede of 1897, pp. 12—13.

Weierstrass begins this paper as follows: "In the introduction to the 'Precis d'une théorie des fonctions elliptiques' of Abel (*Crelle's Journal*, Vol. 4, p. 244; compare Vol. 6, p. 76) the remark is found that the modular function sn u—denoted by Abel as $\lambda\ (u)$—can be expressed as the quotient of two constantly converging series in powers of u whose coefficients are integral functions of the modulus. The present paper contains an attempt to carry out the development of these series as well as of similar series for the remaining modular functions, and at the same time to show how from these series one can go over to the remaining known developments of these functions. It appeared expedient in connection with all the developments to assume only the fundamental properties of the modular functions; further, special care is taken to show the validity of the formulae also for imaginary values of the argument and the modulus".

This paper contains the result of an investigation inspired by the lectures of Gudermann which were attended after the first day by Weierstrass alone. The elliptic functions sn u, cn u, dn u were called modular functions by Gudermann because of

$$u=\int\frac{dz}{\sqrt{(1-z^2)\,(1-k^2z^2)}}$$

being considered by him as the modular integral depending on the modulus k.

Weierstrass proves that $sn\ u=\frac{Al_1}{Al}$, $cn\ u=\frac{Al_2}{Al}$,

$dn\ u=\frac{Al_3}{Al}$, where the Al's are transcendental in u and k^2 in such a manner that the coefficient of each individual term is integral and rational in k^2. Weierstrass calculates the coefficients sometimes up to the 20th power. It is found among other things that the Al's differ from the corresponding θ's of Jacobi's representations,

$$sn\ u=c_1.\ \frac{\theta_1}{\theta},\ cn\ u=c_2.\ \frac{\theta_2}{\theta},\ dn\ u=c_3.\ \frac{\theta_3}{\theta},$$

only by an expotential factor of the form $c\ e^{\lambda u^2}$.

(II)

Darstellung einer analytischen Function einer complexen Veränderlichen, deren absoluter Betrag zwischen zwei gegebener Grenzen liegt" (Published for the first time in *Werke*, Bd. 1, pp. 51—66; the paper was written at Münster in 1841).

"In this paper is found, so far as known, the first rigorous proof, of Cauchy's theorem on the integration between two complex limits, which makes no application of double integrals or integration over a surface; the proof is so arranged that one gets at the same time Laurent's* theorem which became known two years later (*C. R.*, 30 Oct., 1843) because of a note by Cauchy."* Weierstrass's proof assumes that the integrand $f(z)$ should have in a region including the contour a continuous differential coefficient or that everywhere in that region $\left| \frac{f(z+h)-f(z)}{h} - f'(z) \right|$ should be arbitrarily small. In Goursat's second proof, only the existence of $f'(z)$ throughout the region is assumed.

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* Pierre Laurent (1813—1874) was a student of the E'cole Polytechnique, became later Captain in a Geniekorps and was stationed at Havre in 1854. Later he was chief of a battallion in Paris. There is no doubt that the theorem named after Laurent was first discovered by Weierstrass, although the theorem appeared in print first in the papers of Laurent and Cauchy in 1843.

(III)

"Zur Theorie der Potenzreihen (Published for the first time in *Werke*, Bd. 1, pp. 67—74; the paper was written at Münster in autumn 1841).

This paper begins as follows: "Let $F(x) = \sum_{-\infty}^{\infty} A_n x^n$ be a power series of the complex variable with given coefficients. If then r is any definite positive quantity lying inside the circle of convergence of that series, then has the absolute value of $F(x)$, if one gives to the variable x all those values for which $|x| = r$, an upper limit which is denoted by g; and the theorem holds that

$$|A_\mu| \leq g r^{-\mu}$$

for every integral value of μ."

This theorem was first given by Cauchy at Turin in 1832 and was printed in 1841 in *Exercises d' Analyse et de Phys. Math.*, t. 2.

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* *Acta Math*, Vol. 39, p. 27.

(IV)

"Definition analytischer Functionen einer Veränderlichen vermittelst algebraischer Differentialgleichungen" (*Werke*, Bd. 1, pp. 75—84; the paper was written at Münster early in 1842).

The paper contains three articles and the following note is added at the end in the *Werke*, Bd. 1:—"At the time when I wrote this paper, the theorem, established in § 1, had been already discovered and published by Cauchy, of which fact I had not any knowledge. With the present edition of my paper, the contents of which have been used up to now only in connection with my lectures on the theory of functions, in what concerns the article mentioned above, I could not attach myself to Cauchy's representation, for then I would have been compelled to rewrite completely Arts. 2 and 3 which contain important theorems not given by Cauchy, and besides my proof of the chief theorem of § 1 differs from Cauchy's"

What Weierstrass calls the chief theorem is given by him in the beginning of his paper in the following words: "Consider the following system of n differential equations,

$$(1)\quad \left\{ \begin{aligned} &\frac{dx_1}{dt} - G_1\,(x_1, x_2, \ldots x_n) = 0, \\ &\frac{dx_2}{dt} - G_2\,(x_1, x_2, \ldots x_n) = 0, \\ &\frac{dx_n}{dt} - G_n\,(x_1, x_2, \ldots x_n) = 0\ , \end{aligned} \right.$$

in which $x_1, x_2, \ldots x_n$ denote n functions of an independent variable t which are to be determined and $G_1(x_1, x_2, \ldots x_n), \ldots G_n(x_1, x_2, \ldots x_n)$ are given integral rational functions of $x_1, \ldots x_n$; then in a certain neighbourhood of the point $t=0$, n ordinarily converging power-series

$$P_1(t), P_2(t) \ldots P_n(t)$$

admit of being laid down, which, when substituted for $x_1, x_2, \ldots x_n$ in the above differential equations, satisfy them and at the same time take for $t=0$ arbitrarily prescribed values $a_1, a_2, \ldots a_n$.

Cauchy appears to have published an analogous theorem in *Comptes Rendus* of 4. 7. 1842 (See *Oeuvres*, s. 1, t. 7, pp. 5–17). It is interesting to note that in §3 the notion of "continuation" is defined for the first time. It is further clear from this paper that since 1841 Weierstrass had known the notion of analytic function in its complete generality.

(V and VI)

(V). "Bemerkungen über die analytischen Facultäten" (appendix to the Jahresbericht über das Progymnasium zu Deutsch Crone für das Schuljahr, 1842—1843; *Werke*, Bd. 1, pp. 87—103; written at Deutsch Crone in Aug., 1843).

(VI). "Ueber die Theorie der analytischen Facultäten" (*Crelle's Journal*, Bd. 51, 1856; *Werke*, Bd. 1, pp. 153—221; written in 1854).

(VI), which includes, as a part, substantially the whole of (V), contains an introduction and nine articles. In the introduction, the definition of the *analytical faculty*, given by Crelle in 1824 as a function $f(u, x, y)$ of three arguments u, x, y which satisfies the equations,

(1) $f(u,x,y+y')=f(u,x,y).\ f(u+xy,x,y')$,

(2) $f(u,x,1)=u$,

(3) $f(ku,kx,y)=k^y f(u,x,y)$,

where y' is a positive integer and k any arbitrarily increasing quantity, as well as the earlier definitions of Vandermonde (1772), Kramp (1798) and Bessel (1812) and the later definitions of Ohm and Oettinger, have been shown to be open to objections. In Art. 1 it is proved that the general expression for a function $f(u,x,y)$ satisfying (1), (2), and (3) is

$$x^y.\ \frac{\mathrm{Fc}\left(\frac{u}{x}\right).\ \phi\left(\frac{u}{x}+y\right)}{\mathrm{Fc}\left(\frac{u}{x}+y\right).\ \phi\left(\frac{u}{x}\right)}.$$

where

$$\mathrm{Fc}(u)=u.\ \prod_{a=1}^{a=+\infty}\left\{\left(\frac{a}{a+1}\right)^u.\left(1+\frac{u}{a}\right)\right\},$$

and $\phi(u)$ is an arbitrary function of u for which the relation

$$\phi(u+1)=\phi(u)$$

holds. After giving the pecularities of $f(u, x, y)$ and $\mathrm{Fc}(u)$, and a number of interesting expansions, Weierstrass concludes his paper by expressing sin $m\pi$ and cos $m\pi$ by means of faculties ; in fact*,

$$\sin m\pi = \frac{m\pi}{(1,+1)^{-m}\,(1,+1)^{m}},$$

$$\cos m\pi = \frac{1}{(\tfrac{1}{2},+1)^{m}\,(\tfrac{1}{2},+1)^{-m}}.$$

The following remarks of Weierstrass given in the footnote of p. 158 will be found interesting: "The reader will note that the present memoir was already written in the year 1854 and indeed at the explicit request of Crelle, to whom I had communicated my misgivings against his theory of faculties after I had published a part of the contents of the present memoir in the year 1843 (as an addendum to the annual report of the Progymnasium at Deutsch Crone). Although the theory of analytical faculties has not at all in my eyes the importance which mathematicians attached to it in the past times, still I have allowed the memoir to be now reprinted, because it contains much which I believe, even in the present times, will be of use to young mathematicians."

(VII)

"Reduction eines bestimmten dreifachen Integrals" (*Werke*, Bd. 1, pp. 105—109; written at Deutsch Crone 1844).

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* $(u,+x)^y$ is the value of $f(u, x, y)$ when ϕ is a constant.

Taking r for the positive square root $\sqrt{(Ax^2+By^2+Cz^2+Dyz+2Ezx+2Fxy)}$ and u for $fx+gy+hz$, Weierstrass shows that the integral,

$$S \equiv \int\int\int \phi(r,u)\,dx\,dy\,dz,$$

taken for all those values of x,y,z for which r lies between two given limits a,b, ϕ being a finite function of r and u for all such values, can be reduced to the form

$$\frac{2\pi\sqrt{(\lambda)}}{\sqrt{(G)}}\int_{a}^{b} s\,ds \int_{-\frac{s}{\sqrt{(\lambda)}}}^{\frac{s}{\sqrt{(\lambda)}}} \phi(s,\,u)\,du,$$

where G stands for $ABC-AD^2-BE^2-CF^2+2\,DEF$ and λ is a constant that is determined.

(VIII)

"Beitrag zur Theorie der Abel'schen Integrale" (Jahresbericht über das Königl. Katholische Gymnasium zu Braunsberg in dem Schuljahre 1848—49, pp. 3—23; *Werke*, Bd. 1, pp. 111—131; written at Braunsberg in July 1849).

This was Weierstrass's first publication on the subject of Abel's functions; but, as Mittag-Leffler says, "it was natural that this paper should remain almost completely unnoticed. One is not used to find in a school-programme a paper of such importance as Weiertrass's first work on Abelian integrals was." In the introduction to the paper, Weierstrass says:

tical forms". The second chapter is a short one of 9 pages and is headed as follows: "Some general considerations on the representation of single-valued anlytical functions by series".

The year 1854 marks a turning point in the career of Weierstrass. With the appearance of his memoir "Zur Theorie der Abel'schen Functionen" a feeling of the highest admiration for him ran through the whole of the mathematical world. People realized with great surprise that he, a mere school-master at Braunsberg, had worked on quietly on the most difficult problems and abstained from giving out the individual steps in his investigations until the whole approached finality; such self-control was considered to be almost unique in the history of Mathematics. This feeling of admiration had tangible results for Weierstrass. At the suggestion of Richelot, the Philosophical Faculty of the University of Koenigsberg resolved immediately to confer on Weierstrass the doctor's degree *honoris causa* and a deputation headed by Richelot travelled to Braunsberg to hand over the diploma which was given with the following words: "We all have found our master in Herr Weierstrass". All this came to Weierstrass as a surprise but its memory remained to him as dearest and most precious. "Still on his 80th birth-day, by which time Weierstrass had in course of years received all the external honours which the learned world confers on its chiefs, he spoke full of emotion of Richelot's visit and of the conferment of the degree of doctor of Koenigsberg as his most beautiful recollection. However, he added the painful observation: 'everything comes in life rather too late'. He was at the time of the visit of the deputation 40 years old and had during two decades had to spend the greatest part of his powers on school instruction''.[*] With the doctorate came as an inevitable result his receipt from the Government of the status of head teacher at Braunsberg.

During the school year 1855—1856, Weierstrass was on leave in order to complete his scientific work. With the publication of his memoir "Theorie der Abel'schen Functionen" in 1856, high honours came to him: he was appointed to be Professor at the royal (Gewerbe) Polytechnic Institute of Berlin on the 1st July and in the autumn of the same year he was appointed to be Asst. Professor at the Berlin University at the suggestion of Kummer and ordinary Member of the Berlin Academy of Sciences. This posting of Weierstrass was, according to Klein, an event of as striking importance for Mathematics as the establishment of *Crelle's Journal* in 1826.

The address[†] which Weierstrass delivered on his admission to the Academy is so instructive that we reproduce nearly the whole of it in the following lines: "I ought now to explain in a few words what the course of my studies has been up to this time and in what tical forms". The second chapter is a short one of 9 pages and is headed as follows: "Some general considerations on the representation of single-valued anlytical functions by series".

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[†] *Werke*, Bd. 1, pp. 223-226.

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The address† which Weierstrass delivered on his admission to the Academy is so instructive that we reproduce nearly the whole of it in the following lines: "I ought now to explain in a few words what the course of my studies has been up to this time and in what direction I shall direct myself to pursue them. Since the time when under the direction of my highly revered teacher, Gudermann, whom I shall always remember with gratitude, I made acquaintance for the first time with the theory of elliptic functions, this comparatively new branch of Mathematical Analysis has exercised on me a powerful attraction of which the influence on the entire course of my mathematical development has been decisive. This discipline, founded by Euler, cultivated with zeal and success by Legendre but developed in too one-sided a manner, had at that time since a decade undergone a complete transformation because of the introduction of doubly periodic functions by Abel and Jacobi. Those transcendentals, giving to Analysis new quantities of which the properties are remarkable, find also manifold applications in Geometry and in Mechanics and show thereby that they are the normal fruit of a natural development of Science. But Abel, habituated to place himself always at the most elevated point of view, had found a theorem which, comprehending all the transcendentals resulting from the integration of algebraic differentials, had the same importance for them as Euler's theorem had for the elliptic functions. Cut off in the flower of his age, Abel could not himself pursue his grand discovery, but Jacobi made a second discovery not less important : he demonstrated the existence of periodic functions of several variables of which the principal properties were founded on the theorem of Abel and by which he made known the true significance of that theorem. The actual representation of those quantities of an entirely new kind of which Analysis had not until then an example and the detailed study of their properties, became from that time one of the fundamental problems of Mathematics; and as soon as I comprehended the significance and importance of that problem I decided to attempt its solution. It would have been truly foolish if I had thought of solving that problem without having prepared myself by profound study of the existent means which could aid me and without exercising them on less difficult problems. So years passed on before I was able to approach the work proper, and even thereafter, surrounded by unfavourable circumstances, I have been able to make only slow progress. If, however, I was able to obtain some results which the Academy has honoured by its praise, although I have been able to publish them in incomplete shape, I need not expressly state in which direction the objective lies towards which I ought to direct immediately my efforts.

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* See Mittag—Leffler's paper in *Acta Math.*; Vol. 39. p. 51.

† *Werke*, Bd. 1, pp. 223—226.

"However, I shall deem myself fortunate if later on I am able to secure from my studies also some gain for the applications of Mathematics. I have already indicated that I am by no means indifferent to the question whether a theory is suitable for such applications or not. In that connection, I did not fear that one would accuse me by saying that the importance which Mathematics ean claim with the fullest right as pure science would be lowered if I quite particularly held it in high esteem because by means of it alone a truly satisfactory understanding of the natural phenomena can be worked out. Indeed no one can be more ready than I to recognize that one ought not to seek the aim of science outside itself and that it will be not only lowering its dignity but it will be indeed sinning against it if one instead of devoting himself to it with the fullest love were to demand from it only services, will use it only for another subject or for the requirements of life and for that reason will deny oneself the wish to mark out the way for further research and will reject every direction which does not appear to lead to results capable of being used practically. However, I mean that the relation between Mathematics and the study of Nature should be concieved somewhat more deeply than would be the case if,say, the physicist were to recognize in Mathematics only an auxiliary subject or the mathematician were to look on the questions which are placed before him by the physicist as a rich collection of examples for the methods of the mathematician. To-day I cannot dare to follow further this topic which surely is very much at my heart. On the question, however, which I have already indicated, if it is possible to gain, from the abstract theories to which the present day Mathematics appears to apply itself with preference, some thing also immediately useful, I might say that indeed only on a purely speculative way Greek mathematicians had founded the properties of conics long before any one guessed they were the orbits in which the planets move, and that I live in the hope that there will be more functions with properties like those which Jacobi has rendered famous for his Θ function which shows in how many squares every integer can be broken, how one rectifies the arc of an ellipse, and which, I may add, is alone in the position to describe the true law according to which a pendulum oscillates?'

When Weierstrass was appointed to the chair of Assistant Professor at Berlin University, he found the following colleagues already there : Leopold Kronecker (1823—1891) who had independent means and had come to the University in 1855, Eduard Kummer (1810—1893) who had come in the year 1856 as the successor of Dirichlet and W. Borchardt (1817—1880) who had assumed the chief editorship of *Crelle's Journal* beginning with volume 53. It was in this world full of mathematical suggestions that Weierstrass found himself translated from quiet and isolated Braunsberg. During the 40 years that he remained at the University, having become ordinary Professor in 1864, Weierstrass was the chief source of inspiration to a large number of his pupils whose numbers increased from year to year as well as to those who, like some senior men, e.g., Paul du Bois-Reymond, were his intimate friends and admirers but not pupils. Among his pupils may be mentioned, Mittag—Leffler, Schwarz, Koenigsberger, Fuchs, C. Runge, Georg Cantor, Hettner, Knoblauch, Sonja Kowalewsky, Adolph Hurwitz, Max Planck, Hölder,

A. Pringsheim, Schur, Meyer, Mangoldt, Schoenfliès, Wiltheiss, Seliwanoff, Burkhardt, Kneser, R. von Lilienthal, Jules Molk. The influence of Weierstrass on mathematical thought was also greatly due to the various courses, in all 87, which he delivered during the years 1856–1890; a list of these lectures is given at the end of this chapter. After 1890 on account of advanced age he did no lecture work. One of the peculiarities of Weierstrass as a lecturer during the latter years of his life was his inability to write on the board on account of his finding it difficult to move his arm. Every day a pupil was ordered by Weierstrass to write on the board at his dictation and the man chosen deemed it a great honour to be of such assistance, although he himself could not put anything in his note-book and had to depend on his memory. Runge* relates a funny story as to how in his time (1877–1878) Alexander Wernicke who used to be the amanuensis would sometimes dare to correct Weierstrass. Standing by the board, Weierstrass would ask him to write a formula and after Wernicke had written it Weierstrass would wipe it off as he wanted it to be written differently. Wernicke would write as before, Weierstrass would wipe it off and so on. It was very rare for Weierstrass to cite any authority in his lecture and in fact he desired that every statement made by him should have sanction from him and no one else.

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* *Jahresbericht d. d. Math.- Vereinigung*, Bd. 35, p. 176

One of the favourite pupils of Weierstrass, Wilhelm Killing, has given an interesting account of Weierstrass's activities as a teacher at Berlin. In the early years of his Berlin residence, Weierestrass had to give lectures at the Gewerbe Institute for 12 hours a week and in addition at least one private and one public lecture at the University. Also the close touch with his mathematical friends excited to a high degree his research activities. The great strain, caused by his hard work, produced giddiness in Weierstrass on many occasions; and during a lecture on the 16th December, 1861, his giddiness was so great that he could with difficulty be led to a seat. Slowly he improved in the summer of 1862 and it was only in the autumn of that year that he could be allowed to deliver his lectures. The disease, however, never left him completely during the remaining part of his life.

Killing[*] says: "Weierstrass wished for an audience that already possessed good knowledge, who knew that it is not possible even for the most gifted to penetrate into the depth of knowledge without bitter work and who in true enthusiasm for the subject of their choice did not avoid any care or exertion in order to be able to enjoy the noble pleasure of mathematical discovery and research. With every one who attended Weierstrass's lectures, it was necessary that continuous industry at home be combined with the regular attendance at the lectures. He who, however, did not avoid this trouble carried from Weierstrass's lectures such a gain as could not be obtained from any other lectures. What Weierstrass brought out during his lectures, could not be gained from any books; that was almost without exception the sole property of the master. With the enormous mass of his discoveries and the rapidity with which they followed, it was not possible for him to send them to the press in a fully satisfactory form. This rich stuff he communicated, therefore, in his lectures and thereby prevented its loss. Indeed, the knowledge of his most important discoveries was limited in the beginning to a small circle; gradually, however, its fame spread farther and farther. All those who aimed at following the progress of Analysis, those whose aim it was to grasp the individual questions, settled in the proceedings of academies, in connection with the whole system, always sought to obtain the lectures of Weierstrass in their latest forms. Indeed in many cases Professors of Universities left their chairs during the whole of a semester to learn the teaching of the master at his feet. Also foreigners did not remain behind; before the greatness of his discoveries their national pride was suppressed and they recognized openly the superiority of the German mathematician...

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[*] *l.c.*, pp. 15–17.

"If I am to describe the influence which Weierstrass exercised over his pupils, I should not forget his personal intercourse with them. He did not belong to the category of those University teachers who have relations with the audience only in the lecture room; on the other hand he was always eagerly ready to receive them at his residence. Still, it was rather open to any one to join him after the lecture on his way home. Even in personal matters he took heartfelt interest and was also in such matters willingly ready to help by advice. If the question was one of scientific investigations, then he sacrificed for his pupils many hours in order to guide them in their own investigations, to encourage them in investigations already begun, and to bring many improvements in investigations already closed. Many thoughts which later were assumed as the full property of the pupils were first inspired by talks with Weierstrass and ought to have been looked upon with greater right as the property of the master. Indeed, if Weierstrass felt that he could advance the investigations of his pupils by the communication of his own discoveries, he gladly gave away from them without asking whether these that he gave away would go out in the world under his own flag or a foreign one. This was generally known and found its pregnant expression in the frequently repeated joke: Weierstrass rejoices over every thought that is stolen from him only if it is found again with the purloiner. If, however, any of his pupils succeeded in going beyond the ideas of the master in order to find for a problem aspects which he himself had not thought of, then Weierstrass felt a particular pleasure, a pleasure, purer and more real than which a father cannot have over the success of his child."

Although after about 1875 for over 20 years, Weierstrass was almost indisputably the greatest mathematician in Germany, if not in the world, it is by no means true that he had no troubles and no detractors: he had a thorn in his side in the person of his colleague, Leopold Kronecker, about whom Weierstrass writes as follows in a letter* to Sonja Kowalewsky, dated the 24th March, 1885. "Worse it is, however, if Kronecker lends his authority for the view that all, who have worked up to now towards the founding of the theory of functions, are sinners before the Lord. If a wonderful fool like Christoffel says that in 20 or 30 years the present theory of functions will be carried to the grave and the whole of analysis will merge into the theory of forms, then one responds to that by a shrug of the shoulders. When, however, Kronecker makes the assertion which I repeat word for word: 'If to me years and sufficient powers remain, then I myself will still be able show the mathematical world that not only to Geometry but also to the Arithmetic of Analysis can be shown the way and surely the more rigorous. If I cannot do that, then it will be done by those who will come after me and they will recognize the inaccuracy of all those conclusions with which the so-called Analysis works at present', then such an assertion by a man, whose high talents for mathematical research and eminent achievements are sincerely and gladly wondered at by me as well as by his fellow mathematicians, not only shames those towards whom it is directed so that they may accept as error what is the result of their continuous thought and efforts,but it is a direct appeal to the younger generation to leave their leaders".

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* *Acta Math*, Vol. 39, pp. 194–195.

But Weierstrass had also real sources of happiness in the enjoyment of a family life; for, soon after he was appointed at Berlin, he brought his father and his two sisters to live with him. To the father it was destined in his advanced age to see that the greatness of his eldest was more and more recognized and that the expectations that he had formed of him during his youth were far exceeded in middle age. The sisters, specially, enjoyed every honour that was conferred on Weierstrass, who also, although he never sought them, was very pleased with them. Two such honours were the celebration of his 70th birth-day and the celebration of his 80th birth-day. On the first occasion, when at the special request of Weierstrass the gathering was limited to his pupils and their nearest friends, as an outer sign of their reverence his bust in marble together with a gold memorial-medal with his image was presented to him. On the second occasion by the order of the Emperor his picture, painted by Voigtlander, was exposed in the National Gallery; and round the honoured mathematician appeared not only his pupils but also delegates from the Societies and the Universities. . . . .

On the occasion of his 80th birth-day, his health was already much shaken. Even from his 70th birth-day onwards he had been compelled to limit his teaching activities to a few hours; in fact, he was seldom able to bring a course of lectures to its end. "The last years of his life he spent quietly in his house, chiefly engaged in the editing of his works and his lectures, while seated on a roll-chair. One of the two sisters, the devoted attendants of his old age, died in the early part of 1896."* He himself had a lung trouble in the preceding winter which brought about his death on the 19th February, 1896.

I proceed now to give an account of each of the important papers published by Weierstrass after his admission into the Academy of Sciences of Berlin.

(I)

"Ueber das sogenannte Dirichlet'sche Princip"

(Read before the Royal Academy of Sciences of Berlin on 14-7-1870; *Werke*, Bd. 2, pp. 49—54).

In order to prove the theorem: "that for any given finite connected space t there exists always one, and only one, function w which changes everywhere in t in a continuous manner together with its first partial differentiel coefficients, has on the boundary of t everywhere arbitrarily prescribed (but continuously varying) values and satisfies everywhere inside t the equation

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}=0\text{"},$$

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[*] The substance of this and preceding paragraphs is from Killing's speech.

Dirichlet made use of the following doctrine which was later on called by Riemann Dirichlet's principle: Of all the functions *u*, which together with their first partial differential coefficients change in a continuous manner everywhere in *t* and, on the boundary of *t*, take prescribed values, there must exist one (or several) for which the integral

$$U=\int\left\{\left(\frac{\partial u}{\partial x}\right)^2 \quad \left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right\}dt,$$

extended over the whole space *t*, has its minimum value.

Weierstrass says that the existence of such a minimum does not always follow, "but it can be only asserted that for the expression in question (*i. e.* U) a lower limit exists which it can approach arbitrarily closely but which it may not be possible to attain to" with a *u* of the desired type. In fact, the lower limit may be attained by, say u_0, which no longer possesses the properties of continuity and differentiability presupposed for the *u*' s. This is shown by Weierstrass by an example.

Riemann recognized the justice of Weierstrass's criticism but said, as reported by Weierstrass to Klein, "he (Riemann) had used the Dirichlet's principle as a convenient means which was readily available—notwithstanding the criticisim his existence theorem was true". The existence theorem referred to is the one given in the beginning of the review of this paper.

(II)

"Neuer Beweis eines Hauptsatzes der Theorie der periodischen Functionen von mehreren Veränderlichen",

(Berliner Monatsberichte 1876; *Werke*, Bd. 2, pp. 55—70)

The theorem proved is the following: A single-valued or m-valued function, where m is finite, of p variables can possess a system of infinitely small periods only in the case in which the function admits of being expressed as a function of *less* than p linear combinations of its arguments. This theorem was first proved in this generality by Riemann (*Crelle's J.*, Bd. 71, 1870; *Werke*, 1876, p. 276). That a single-valued function of p variables cannot have more than 2 p independent periods without possessing infinitely small periods, was first proved by Jacobi (*Crelle's J.*, Bd. 13, 1835; *Werke*, Bd. 2, p. 233) for $p=1$, and for arbitrary p by Hermite (*Crelle's J.*, Bd. 40, 1850, p. 310; *Oeuvres*, *t.* 1, p. 158).

In the note of 1886 at the end of the paper Weierstrass says:

"The question under what conditions the theorem remains valid for an analytic function $f(u_1, u_2, . . u_n)$ also in the case in which it is so constituted that, corresponding to each system of values of the variables $u_1, u_2 . . . u_n$, there are *infinite* number of values of $f(u_1, u_2, ... u_n)$, has not as yet been settled. That there are functions for which the theorem does not hold is known. Herr Casorati has already shown how analytic functions of one argument admit of being defined in very simple manner so that their periods cannot be derived from only two of them". See in this connection a paper by Casorati in *Acta Math.*, t. 8.

(III)

"Ueber continuirliche Functionen eines reellen Arguments, die für keinen Werth der letzteren einen bestimmten Differentialquotienten besitzen." (Read before the Academy of Berlin on 18-7-1872; *Werke*, Bd. 2, pp. 71–74).

In this is given the continuous, but nowhere differentiable, function

$$\sum_{n=0}^{\infty} b^n \cos (a^n x\pi),$$

where a is an odd integer and b is a positive constant less than 1.

In the introduction, Weierstrass says that, in 1861 or even earlier, Riemann spoke to his pupils that a continuous but nowhere differentiable function can exist and gave as example

$$\sum_{1}^{\infty} \frac{\sin (n^2 x)}{n^2}.$$

But no proof of non-differentiability has been found in Riemann's papers and such a proof seems to be difficult* to give. However, continuous and nowhere differentiable functions can be easily constructed and the proof of their property can be most easily given. Then he proceeds to give his function.

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* In a letter to Du Bois-Reymond, dated the 23rd November, 1873, Weierstrass says that Riemann was reported to have said that the proof would come from elliptic functions. See *Acta Math*, Vol. 39, p. 199.

(IV)

"Zur Theorie der eindeutigen analytischen Functionen" (*Berliner Abhandlungen*, 1876; *Werke*, Bd. 2, pp. 77—124).

The object of the memoir is thus stated by Weierstrass: It has been shown "that one needs to know of a function $f(x)$ only this, that it is a single-valued function *without* an essential singularity, in order to be sure that it is representable as the quotient of two integral rational functions of x (of which one can also reduce itself to a constant); in other words, it has been shown that by the two assumed properties of the function also the arithmetical dependence of its value on the value of the independent variable is conditioned and determined. Thereby is the question brought near, whether for the single-valued functions with a finite number of essential singularities something similar holds—*whether it is possible to form arithmetical expressions composed of the variable x and undetermined constants, which represent all the functions of a definite class and only these functions.*

"In the present memoir is completely settled this question, in which a problem is announced which belongs to the elements of the theory of functions in general and at the same time well-limited."

The memoir is divided into 8 articles which are respectively headed as follows:—Introduction; About the theory of integral single-valued functions of a variable; Single-valued functions of x with one essential singularity; An auxiliary theorem; Single-valued functions of x with a finite number of (essential or non-essential) singularities; Single-valued functions of x, which possess n essential singularities, and have at every other point a finite value different from 0; Single-valued functions of x with n essential and an arbitrary number of non-essential singularities; Behaviour of the investigated function in the neighbourhood of one of its essential singularities.

In this famous memoir are given two well-known theorems of Weierstrass: Art. 2 gives the factor-theorem and Art. 8 the theorem about the behaviour of a function in the neighbourhood of an essential singularity.

(V)

"Untersuchungen über die $2r$-fach periodischen Functionen von r Veränderlichen" (*Crelle's J.*, Bd. 89; *Werke*, Bd. 2, *pp.* 125—133). Weierstrass corrects in this paper the theorem which he had given in 1869 (*Werke*, Bd. 2, pp. 46—48) for single-valued functions of n independent variables and having $2n$ periods. The theorem is a generalization of Liouville's theorem, that two elliptic functions having the same periods are connected by an algebraic relation. At the end of the paper, Weierstrass announces that every single-valued function of the kind considered admits of being expressed in terms of a Θ-function of r variables.

(VI)

"Einige auf die Theorie der analytischen Functionen mehrerer Veränderlichen sich beziehende Sätze" (*Werke*, Bd. 2, pp. 135—188).

The paper is divided into six articles and some of the headings of these articles are the following: Preparatory theorem; About the theory of single-valued analytical functions of an arbitrary number of variables; About the theory of integral single-valued functions of an arbitrary number of variables; About the development of n—ply periodic integral functions of n variables.

The preparatory theorem is the following:—Let $F(x, x_1, x_2, \ldots x_n)$ be a function of $x, x_1, \ldots x_n$, given in the form of an ordinary power series; let the function be 0 when the variables are all zero. Then there exist an infinite number of systems of values,

$$x, x_1, \ldots x_n,$$

belonging to the convergence-region, which satisfy the equation

$$F(x, x_1, \ldots x_n) = 0.$$

Defining a power series $P_1(x_1, \ldots x_n)$ as *divisible* by another power series $P_0(x_1 \ldots x_n)$ when a third power series $P_2(x_1, \ldots x_n)$ admits of being determined so that

$$P_1(x_1, \ldots x_n) = P_0(x_1, \ldots x_n) P_2(x_1, \ldots x_n),$$

the necessary and sufficient condition is investigated for the divisibility of P_1 by P_0. Then the necessary and sufficient condition is found in order that two given power series P_0 and P_1 which vanish at the same point $(x_1 = 0, \ldots x_n = 0)$ are both divisible by a third series of the same property. Weierstrass defines essential and non-essential singularities and proves that for $n > 1$ the latter are of two quite different types. It is proved that the totality of the points where $f(x_1, .. x_n)$ behaves like a rational function is a $2n$— dimensional continuum which is bounded by the essential singular points.

(VII)

"Ueber einen functionentheoretischen Satz des Herrn G, Mittag-Leffler" (*Berliner Monatsbericht*, 1880; *Werke*, Bd. 2, pp. 189—199).

In this paper, Weierstrass gives a simpler proof of the theorem which Mittag-Leffler published in 1877 in connection with the memoir discussed under (IV) above, and also takes the opportunity "to complete the results of that memoir in several essential points".

(VIII)

"Zur Functionenlehre" (*Berliner Monatsbericht*, 1880 and 1881; *Werke*, Bd. 2, pp. 201—233).

Weierstrass introduces the memoir thus: "In the following I communicate some investigations, relating to infinite series having for its terms rational functions of a variable, which have the aim to make clear certain peculiarities—not as yet observed as far as I know—which such series can offer and whose knowledge is of importance for the theory of functions."

Some of the theorems given in this important memoir may be enumerated here :—

(1) The series $\sum\limits_{n=0}^{\infty} f_n(x)$ converges uniformly in a given part B of its region of convergence if quantities $g_0, g_1, g_2,\ldots$ can be found so that at every point of B

$$|f_n| < g_n$$

and

$$\sum_{n=0}^{\infty} g_n \text{ is convergent.}$$

[Weierstrass's Test for uniform convergence].

(2) If a series converges uniformly in the neighbourhood of every point, which lies in the inside or on the boundary of a given connected region, then the series converges uniformly in the whole region.

(3) If a series $\sum\limits_{0}^{\infty} f_n(x)$ is such that, in the neighhood of every point in the inside of its region of convergence, it converges uniformly, then the series defines in general in each part of that region a single-valued branch of a monogenic analytic function, and in particular it represents such a function completely.

(4) If a series whose members are power series in $(x-x_0)$, *i. e*, a double series, converges uniformly in a circle, then it admits of being written down as a simple power series, whose members are obtained by the addition of the terms of the same powers in the original

series and the simple power series converges in the same circle. [Double-series Theorem].

(5) It is possible to construct a function

$$F(x)=\sum_{0}^{\infty} f_n(x)$$

so that the series shall have its region of convergence consisting of n parts (A_1, A_2,... A_n), and $F(x)$ shall equal $F_1(x)$ in A_1, $F_2(x)$ in A_2 ...,$F_n(x)$ in A_n, where $F_1(x)$. $F_2(x)$,...are *arbitrarily chosen functions* which represent single-valued monogenic functions everywhere with the exception of some points.

An example of a series whose region of convergence consists of two parts in one of which the sum is 1 and in the other -1 was given by Weierstrass. Such a series is $\frac{1+x}{1-x}+\frac{2x}{x^2-1}+\frac{2x^2}{x^4-1}+\frac{2x^4}{x^8-1}+\dots$ which converges everywhere with exception of the circumference $|x|=1$ and equals 1 or -4 according as $|x|<$ or >1.

In this memoir Weierstrass also gives the series $\Sigma b^n x^{a^n}$, where b is a positive quantity less than 1 and a is an odd positive integer, as having a natural boundary when $ab>1+\frac{3\pi}{2}$.

(IX)

"Zur Theorie der elliptischen Functionen" (2 papers with the same title, published in the *Berliner Berichte*, one in 1382 and the other in 1883; *Werke*, Bd. 2, pp. 245 – 309).

The results of these two papers appeared in the *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen. Nach Vorlesungen und Aufzeichnungen des Herrn K. Weierstrass bearbeitet und herausgegeben von H. A. Schwarz.* (1st edition 1881—83; 2nd edition 1893).

In the first paper the results are given under the headings: Derivation of the relations which hold among the functions

$$\sigma(u \mid \omega, \omega'), \sigma_1(u \mid \omega, \omega'), \sigma_2(u \mid \omega, \omega'), \sigma_3(u \mid \omega, \omega')$$

and their partial differential coefficients with respect to u, ω, ω'; Use of the differential equations given above for the expansions of the functions, σ, σ_λ.

The object of the second paper is stated as follows; In order to express the elliptic function $sn\,(u, k)$, $cn\,(u, k)$, $dn\,(u, k)$ by means of Jacobian theta-functions $\theta(x, q)$, $\theta_1(x, q)$, $\theta_2(x, q)$, $\theta_3(x, q)$, one has to solve the problem of finding a q satisfying for a given value of k the equation

$$\sqrt{(k)} = 2q^{\frac{1}{4}}, \quad \frac{1+q^2+q^6+q^{12}+\dots}{1+2q+2q^4+2q^9+\dots} \qquad (1)$$

In the paper of Jacobi, "Theorie der elliptischen Functionen aus der Eigeinschaften der Thetareihen abgeleitet" this problem is solved for the case when k is a real quantity lying between 0 and 1. "I shall show now how one, with the means used by Jacobi in the above-mentioned paper, can determine for any complex

k, all the values of q satisfying (1), and indeed by means of a series, which not only because of its rapid convergence admits of a comfortable calculation of the value of q corresponding to a given numerical value of k, but also can serve in finding the characteristic properties of q considered as a function of the variable quantity k."

(X)

"Zur Theorie der aus n Haupteinheiten gebildeten complexen Grössen" (Read before the Royal Society of Sciences of Göttingen on 1-12-1883; *Werke*, Bd. 2, pp. 311—339).

Inspired by Gauss's investigation of the question, whether for complex quantities with n chief units the fundamental operations of addition, multiplication subtraction and division admit of being so defined that the theorems which are valid for the so-called real quantities, viz., those corresponding to $n=1$, retain their validity, and the conviction of Gauss that this was impossible for $n>2$; Weierstrass undertakes in this paper the study of the Arithmetic of the general complex quantitics with n chief units and proves that this Arithmetic "cannot lead to any result which is not without difficulty derivable from the Arithmetic of the complex quantities with one or two chief units."

(XI)

"Zu Lindemann's Abhandlung; „Ueber die Ludolph'sche Zahl"". (*Berliner Bericht*, 1885; *Werke* Bd. 2, pp. 341—362).

Weierstrass introduces his paper as follows: "For the result of the investigations of Herr Lindemann, communicated in the above-mentioned paper, (See *Berliner Bericht* of 3-12-1885; also *Math. Ann.*, Bd. 20), viz. the hitherto vainly attempted proof that the number π is not an algebraic number and consequently the quadrature of the circle cannot be carried out in a constructive way, in the widest circles such lively interest has been noticed that I believe it would be welcome to many mathematicians to have, as is given in the present paper, as elementary as possible a proof, of Lindemann's theorem, which is based on well-known theorems alone."

(XII)

"Ueber die analytische Darstellbarkeit willkürlicher Functionen reeller Argumente" (*Berliner Bericht*, 1885; *Werke*, Bd. 3, pp. 1–37).

Among the theorems proved in this very important memoir are the following:—

(1) Every function $f(x)$ which is continuous in a closed interval can be represented there in a uniformly and absolutely convergent series of polynomials. (Weierstrass's Polynomial Theorem).

(2) Every continuous function of the period 2π can be expanded in a uniformly and absolutely convergent series of finite trigonometrical polynomials, *i. e.*, expressions of the form

$$a_0+(a_1 \cos x+b_1 \sin x)+(a_2 \cos 2x+b_2 \sin 2x)$$
$$+\ldots+(a_r \cos rx+b_r \sin rx),$$

where the a's and b's are arbitrary constants.

(3) Any arbitrarily given continuous function can be approximated to by a polynomial which will differ, in any given interval, from the function by an arbitrarily chosen small quantity.

(4) If $f(x)$ is a single-valued function which is continuous in an interval $(-c, c)$, then at any point x inside the interval

$$f(x) = \lim_{n=\infty} \frac{1}{2c} \int_{-c}^{c} f(t)\, \chi\left(\frac{x-t}{c}\pi, n\right) dt,$$

where

$$\chi(v, n) \text{ denotes } 1 + 2 \sum_{r=1}^{n} \{n\}^{r^2} \cos r v,$$

$\{n\}$ standing for $(n+1)^{-\frac{m}{(n+1)^2}}$ where $m > 1$.

Weierstrass considers the question of the extension, to discontinuous functions and to functions of more than one variable, of the above results.

(XIII)

"Untersuchungen über die Flächen deren mittlere Krümmung überall gleich Null ist" (*Berliner Bericht*, 1866; *Werke* Bd. 3, pp. 39—52). The introduction begins as follows: "I have closely occupied myself with the theory of these surfaces, the so-called minimal surfaces, particularly for this reason that the theory, as I shall show, is most intimately connected with the theory of the analytic functions of a complex variable, so that to every such function a surface of the kind in question corresponds, and conversely."

If $F(s)$ denote a function of the complex variable, then it is proved that

$x=$ real part of $\{(1-s^2)F''(s)+2sF'(s)-2F(s)\}$,
$y=$ „ „ „ $\{i(1+s^2)F''(s)-2isF'(s)+2iF(s)\}$,
$z=$ „ „ „ $\{2sF''(s)-2F'(s)\}$,

give a minimal surface.

(XIV)

"Allgemeine Untersuchungen über $2n$— fach periodische Functionen von n Veränderlichen" (*Werke*, Bd. 3, pp. 53–114).

Weierstrass says in the last two paragraphs of the introduction: "Every single-valued function of n arguments, which, for finite values of these arguments, possesses the character of a rational function and is at the same time $2n$— ply periodic, originates in the described manner from a Θ—function of the same variables."

"The establishment of this theorem forms the subject of the present memoir in which, in addition to the theorems belonging to the general theory of analytic functions, the essential properties of Abelian integrals will be assumed to be known".

(XV)

"Zur Theorie der Jacobi'schen Functionen von mehreren Veränderlichen" (*Berliner Bericht*, 1882;

Werke, Bd. 3, pp. 155—159).

In this paper, Weierstrass generalizes the equation

$$\begin{aligned}&\sigma(u+u_1).\ \sigma(u-u_1).\ \sigma(u_2+u_3).\ \sigma(u_2-u_3)\\+&\sigma(u+u_2).\ \sigma(u-u_2).\ \sigma(u_3+u_1).\ \sigma(u_3-u_1)\\+&\sigma(u+u_3).\ \sigma(u-u_3).\ \sigma(u_1+u_2).\ \sigma(u_1-u_2)=0,\end{aligned}$$

first given by him in his lectures in 1862, and obtains a similar equation involving sigma functions of several variables, each of such sigma functions being defined in terms of the Θ_1—function of the variables in the same manner as in the case of one variable.

(XVI)

"Ueber eine die Raumcurven constanter Krümmung betreffende von Delaunay herrührende Aufgabe der Variationsrechnung" (*Werke*, Bd. 3, pp. 183—219).

Weierstrass says in the introduction to the paper: "In the following, first the question about the representation of the most general integrals of the differential equation of the problem will be treated; it will be found that the determination of the sought curve is possible in the most general case with the help of elliptic transcendentals." The problem, referred to, is that treated first by Delaunay in 1843: Of all curves with prescribed constant curvature, which pass between two given points continuously and without sudden alteration of direction, to determine that whose length is a maximum or minimum.

(XVII)

"Analytische Bestimmung einfach zusammenhängender Minimalflächenstücke deren Begrenzung aus geradlinigen, ganz im Endlichen liegenden Strecken besteht" (*Werke*, Bd. 3, pp. 221—239).

Weierstrass says: "The problem which is treated is the following: To determine analytically a simply-connected piece M of a minimal surface whose boundary L consists of n straight lines, lying entirely in the finite part of space, which form a simple closed and unknotted line." It is proved that the problem reduces to the determination of particular integrals $G(u)$, $H(u)$ of a linear homogeneous differential equation of the second order having for its coefficients rational functions of the independent variable u.

(XVIII)

"Ueber eine besondere Gattung von Minimalflächen" (*Berliner Bericht*, 1867; *Werke*, Bd 3, pp. 240—248).

The object of the paper is to give a *periodic* minimal surface. Its equation is obtained in the form

$$\sum_{l,m,n} \left\{ (-1)^{ln+m} e^{\chi(l,\, m+\frac{1}{2},\, n)} \times \sin\left(\frac{lx'}{a} + \frac{(m+\frac{1}{2})y'}{\beta} + \frac{nz'}{\gamma} \right) \right\} = 0,$$

where x', y', z', denote homogeneous linear functions of x, y, z, which may be regarded as the co-ordinates of the point (x, y, z) referred to a definite system of co-ordinate axes, generally oblique; a, β γ are positive constants; l, m, n are integers of which each has to take in the summation all the values from $-\infty$ to ∞, and

$\chi(l, m+\frac{1}{2}, n)$ is a homogeneous function of the second degree in $l, m+\frac{1}{2}, n$ with real coefficients of the property that the function takes only negative values. The periodicity of the surface is clear from the form of the equation. About the end the paper contains some wrong assertions.

(XIX)

"Zur Determinantentheorie" (*Werke*, Bd. 3, pp. 271—287). If n^2 quantities are arranged in n lines, each line containing n of them, and $a_{\alpha\beta}$ denote the βth element of the αth line, then the problem is studied, to form a function A of the quantities $a_{\alpha\beta}$ such that

(1) A shall be an integral linear homogeneous function with respect to the elements of each line; (2) A shall change only its sign when two lines are interchanged; (3) A shall take the value 1 when $a_{11}=a_{22}=\ldots=a_{44}=1$ and all the other elements are equal to 0.

It is proved that such a function A admits of being uniquely determined and is identical with the function commonly called the determinant $\left| a_{\alpha\beta} \right|$.

The fundamental theorems of the theory of determinants are deduced from the above-mentioned three properties of A.

(XX—XXII)

Three papers on quadratic forms:—

(XX) "Ueber ein die homogenen Functionen zweiten Grades betreffendes Theorem". (*Werke*, Bd. 1, pp. 233—246).

(XXI). Supplement to the above with the same title (*Werke*, Bd. 3, pp. 139—148).

(XXII). "Zur Theorie der bilinearen und quadratischen Form" (*Werke*, Bd. 2, pp. 19—44).

In (XX) which was published in 1858, Weierstrass establishes the following therem :—

"If $\phi(x_1, x_2, \ldots x_n)$, $\psi(x_1, x_2, \ldots x_n)$ are integral homogeneous functions of the second degree in the n variables $x_1, x_2, \ldots x_n$ with real coefficients, and, further, if the first is so constituted that for real values of $x_1, x_2, \ldots x_n$ it always retains the same sign and vanishes only in the case when they are all zero ; then n homogeneous linear functions $u_1, u_2, \ldots u_n$ of $x_1, x_2, \ldots x_n$ with real cœfficients and n real constants $s_1, s_2, \cdots s_n$ always admit of being so determined that

$$\pm \phi(x_1, x_2, \ldots x_n) = u_1^2 + u_2^2 + \ldots + u_n^2,$$

$$\psi(x_1, x_2, \ldots x_n) = s_1 u_1^2 + s_2 u_2^2 + \ldots + s_n u_n^2.$$

(In the first equation the upper or the lower sign is to be taken according as ϕ, for real values of $x_1, x_2, \ldots x_n$ which are not all zero, is always positive or always negative).

(XXII), which was published in 1868, contains among other things the above theorem as a corollary of the general theory of the simultaneous transformation of bilinear or quadratic forms given there.

(XXI), published in 1879, supplements in certain respects (XX) and (XXII).

The importance of Weierstrass's lectures has already been mentioned on p. 248 of this volume; their special importance will be obvious when we note that he was reluctant to give out to the world his original thoughts in print, so that most of them are to be found in his lectures. As Klein * says: "*Weierstrass's Lectures* are for us for this reason so particularly important that he himself printed so little. He had—and this is decidedly a very remarkable phenomenon in these 'times of Guthenberg'—a repugnance towards printing-ink which was based on principle. So he never allowed his lectures to be autographed but desired that they should be copied by hand. It was in those days the custom in Berlin to copy, quite according to plan, what one wished to take from Weierstrass's course of lectures. These copies have been circulated also in foreign countries so that they have exercised a considerable influence on the progress of our science...I might only mention the usual rotation of the courses: Analytical Functions, Elliptic Functions, Applications of Elliptic Functions, Hyperelliptic or Abelian Functions,...According to my recollection—I came to Berlin in 1869 and was there in 1869 – 1870—Weierstrass's position was that of an absolute authority whose teaching the audience accepted as an indisputable standard, often without having quite understood it in a deeper sense. A doubt could not spring up and a check was already for this reason almost impossible that Weierstrass cited extraordinarily little authority. He had placed for himself the aim, to give in his lectures a system of well-ordered and connected thoughts. Therefore he began with a methodical construction from below upwards and, striving for his ideal of perfect completeness, he arranged the course in such a manner that in the sequel he needed to refer back to himself alone ...... Gradually Weierstrass won appreciation as a matchless authority in the whole scientific world" so that Hermite could say with truth: "Weierstrass est notre maitre á tous."

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* *l. c.*, pp. 283—284.

Weierstrass agreed very late in his life to the bringing out of his lectures in his *Mathematische Werke* and the work of bringing them out was entrusted in the case of each course of lectures to one or two special pupils or the pupil of such pupils. Thus the course on Abelian transcendentals was entrusted to G. Hettner (1854-) and J. Knoblauch (1855-) two of the most obedient pupils of Weierstrass, that on the theory of elliptic functions to J. Knoblauch and that on the application of elliptic functions to Rudolf Rothe, who also brought out the course on the Calculus of Variations. These are the only courses brought out up to now.

We proceed to outline briefly each of these four courses.

(I)

Vorlesungen über die Theorie der Abelschen Transcendenten (*Mathematische Werke*, Bd. 4, Berlin, 1902).

The lectures cover the whole of the 4th volume of Weierstrass's mathematical works and occupy 631 quarto pages. The volume begins with an introduction of ten pages dealing with the fundamental notions of the Abelian transcendentals and historical remarks, and this is followed by 34 chapters, divided into three parts with the headings : Algebraical foundation of the theory (pp. 11—246), the Abelian integrals (pp 247—438; and the Abelian functions (pp. 439—624). The volume ends with an index (pp. 625—631).

The headings of the chapters are as follow :—

(1) The algebraical frame (Gebilde). (2) Rational functions of the pair (x,y). (3) Representation of a rational function of the pair (x,y) by the function $H(xy,x'y')$. (4) The ρ linearly independent functions $H(xy)_a$. (5) First kind of the calculation of the range of an algebraical frame. (6) Second kind of the calculation of the range of an algebraical frame. (7) The formation of the functions $H(xy)_a$. (8) The formation of the function $H(xy,x'y')$. (9) The formation of a rational function of the pair (xy) with a single infinity. (10) The characteristic properties of the algebraic functions. (11) Introduction to the theory of Abelian integrals. (12) The integrals of the first, second and third kinds, (13) The periods of the

logarithm. (14) The complete integral of the algebraic differentials. (15) The function $\Omega\ (xy)$. (16) The periods of the Abelian integrals of the first and second kinds. (17) The periods of the hyper-elliptic integrals. (18) The functions E $(xy)_\beta$ and E' $(xy)_\beta$. (19) The functions $\Omega\ (xy;x_1y_1,x_0y_0)$ and E $(xy;x_1y_1,x_0y_0)$. (20) Representation of the integral of the third kind by logarithms of the E—functions. (21) The Abelian Theorem. (22) Definition of the Abelian functions. (23) The functions E $(x\ y\ ;\ u_1,\ u_2,\ldots u_\rho)$ and J $(u_1,\ u_2,\ldots u_\rho\ )_\beta$. (24) Representation of the function E $(xy,x_0y_0;u_1,\ldots u_p)$ by a quotient of two constantly converging power-series. (25) Introduction of the function $\Theta\ (u_1,\ldots u\rho)$. (26) The determinant $|\ \omega_{a\beta}\ |$. (27) Introduction of the function $\theta\ (v_1,\ldots v\rho)$. (28) Representation of the function $\theta(v_1,\ldots v\rho)$. (29) Proof for the determinant $|\ \omega_{a\beta}\ |$ being different from zero. (30) The general theta—functions. (31) Representation of the Abelian integrals of the second and third kinds with the help of Θ—functions. (32) The system of values for which the Θ—function vanishes. (33) Representation of Abelian functions. (34) Relations among the products of theta—functions.

The contents of the volume are taken chiefly from the copy prepared, by Hettner and Knoblauch, of the lectures on Abelian functions delivered in the winter—semester of 1875—76 and the summer semester of 1876; the lectures of the earlier and later years being used to a little extent for completing the volume. As stated in the preface, "Weierstrass himself took notice of a small part of the contents: when he died on the 19th February, 1897, the printing had advanced only up to the 18th forma". But, according to Mittag-Leffler*, Weierstrass "was far from being satisfied with the work of the editors; the work lacked the exactitude, the transparent clarity and the unshatterable logical sequence which he always required in his works". Mittag-Leffler opines that neither Hettner nor Knoblauch was a mathematician of such importance as to discharge his duty as editor with complete satisfaction; although they took nearly 10 years to bring out the volume.

(II)

Vorlesungen über die Theorie der elliptischen Functionen (*Math. Werke*, Bd. 5, Berlin, 1915).

The whole volume is devoted to the lectures and contains 327 pages; the volume begins with an introduction of three pages illustrating the notions of *double periodicity, addition-theorem, elliptic integrals* and their *inverse functions*. The introduction is followed by 34 chapters with the respective headings: transformation of the differential $\frac{dx}{\sqrt{R(x)}}$, integration of the differential equation

$$\left(\frac{dx}{du}\right)^2 = R(x)$$

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* *Acta Math*. Vol. 39, p. 53.

by expansion in series, the function $\wp(u)$, the function $\sigma(u)$, the partial differential equation of the σ—function, solution of the equation $\wp(u) = s$ by series, determination of all the solutions of the equation $\wp(u)=s$, fundamental formulae of the theory of the σ—function, the periods of the $\wp$—function for real invariants, the functions $\sigma a(u)$ and the σ—quotients, the differential equations of the σ—quotients, representation of the σ—function by an infinite product, transformation of the infinite product for the σ—function, representation of elliptic functions by means of the σ—function, representation of elliptic functions by the $\wp$—function, representation of the functions $\sigma_a(u)$ by infinite products, further transformation of the product expressions for the σ—functions, the four theta—functions, the general theta-function, the theta-function with two parameters and transformation-formulae for the $\wp$—and σ—functions, relations among σ—functions of several arguments, the addition-theorems of the σ—quotients and relations among theta-functions of several arguments, the multiplication theorem of the $\wp$—function, the multiplication theorem of the σ—quotients, the elliptic integrals, the addition-theorems of the integrals of the first, second and third kinds, formulae for the calculation of the periods, determination of a primitive pair of periods of the $\wp$—function for arbitrary quantities e_α, determination of u from the equation $\wp(u)=s$, application of the formulae of the 13th and 19th chapters to the case of real invariants, transformation of the elliptic functions, transformation of special functions, about the transformation of the $\wp$—function, the transformation of the second order.

The book ends with an index; in the preface it is stated that with the exception of Chapters 1—9, 12 and 13, which are based on a manuscript dictated by Weierstrass to F. Mertens in 1863, practically the whole of the volume is from the copy, prepared by Knoblauch, of a course delivered by Weierstrass in the winter-semester of 1874–75.

(III)

Vorlesungen über Anwendungen der elliptischen Functionen (*Math. Werke*, Bd. VI, Berlin, 1915).

This volume, devoted entirely to the lectures, covers 355 pages, divided into 31 chapters arranged in nine parts; the headings of the various chapters being respectively: representation of the curved surface of an oblique circular cone by an elliptic integral of the third kind, calculation of the elliptic integral

$$\int_0^\omega \frac{\wp'(u_0)}{\wp(u)-\wp(u_0)}\,du,$$

elliptic coordinates, determination of a triangle formed by the lines of curvature of the ellipsoid, unique determination of the logarithms occuring in the formula for the surface of the ellipsoid, calculation of the total surface of the ellipsoid, the potential of a homogeneous body bounded by an ellipsoidal surface, determination of the integral

$$\int_0^{2\pi} d\psi \Big/ \sqrt{\{a\cos^2\psi + b\sin^2\psi + c + 2a'\sin\psi + 2b'\cos\psi + 2c'\cos\psi\sin\psi\}},$$

determination of the potential of the surface of an ellipse loaded with matter uniformly, determination of the interval for the value of the integral

$$\int_{x_0}^{x_1} dx \Big/ \sqrt{(R(x))}$$

and transformation of the elliptic differential into the Legendre—Jacobi normal form, linear transformation of the elliptic differential $dx/\sqrt{R(x)}$ into the Legendre—Jacobi normal form, transformation of the second degree of the elliptic differential into the Legendre normal form, general integration of the elliptic differential equation, special integrals of the elliptic differential equation, integration of the elliptic differential equation when its coefficients are all real, real-valued integrals of the elliptic differential equation with real coefficients, the lemniscate functions and the division—equation of the lemniscate, the roots of the lemniscate division—equation and the division of the lemniscate into five equal parts, the solution of the lemniscate division–equation, on the complex multiplication of the elliptic functions, determination of the coordinates of the spherical pendulum, continuation (the plane pendulum), preliminary investigations and the laying down of the Eulerian differential equation, determination of three direction—cosines by means of the sigma—quotients, continuation (determination of the remaining direction—cosines), the motion of the instantaneous rotation—axis, representation of the nine direction—cosines by four parameters, on the motion of a rigid body under gravity round a fixed point, representation of the cartesian coordinates of a point of a geodesie line on an ellipsoid of revolution by elliptic functions, determination of the arc—length of the geodesic lines on the ellipsoid of revolution.

To a great extent, the contents of this volume are taken from the copy, prepared by Hettner, of the course of lectures delivered by Weierstrass in the summer of 1875.

(IV)

Vorlesungen ueber Variationsrechung (*Werke*, Bd. VII, Berlin, 1927).

The volume of 324 pages is devoted entirely to the lectures which are given in the first 318 pages followed by remarks and an index on the remaining pages. The introduction occupies the first two pages and then follow 31 chapters, which are grouped under two parts, viz. the first part, of six chapters, on the theory of maxima and minima of functions of one or several variables and the second part on Calculus of variations.

The chapters are headed as follow: Explanations and laying down of the necessary conditions; auxiliary theorems from the theory of quadratic forms, sufficient conditions for the existence of a maximum or minimum of a function of several variables ; maxima and minima with auxiliary conditions ; further theorems on quadratic forms ; closing remarks on the theory of maxima and minima ; applications ; examples about the theory of maxima and minima ; introductory remarks, on the surfaces of revolution of least area ; continuation, representation of the coordinates as functions of a parameter, other special problems of the Calculus of Variations ; properties of the function F x, y, x', y') ; the first variation and the differential equation $G=0$; the continuity of $\frac{\partial F}{\partial x'}$ and $\frac{\partial F}{\partial y'}$, other forms of the differential equation $G=0$; some examples about the 10th and 11th chapters ; the second variation ; continuation, sign—condition of the function F_1 ; investigations of Jacobi ; the conjugate points ; proof of Jacobi's criterion ; region of validity of the conditions hitherto found, examples ; continuation, auxiliary theorems of the theory of functions ; geometrical significance of the conjugate points ; on the body of revolution whose boundary surface offers to air the least resistance ; the fourth necessary condition ; proof that the fourth condition is also a sufficient one ; complementary remarks, examples ; isoperimetrical problems ; examples of isoperimetrical problems ; sufficient conditions ; establishment of the suppositions made in the preceding chapters ; continuation and closing observations ; examples ; variation of the end-points and unfree variations.

The lectures have been brought out by R. Rothe, who states in the preface that, for his purpose, he made use of various copies; viz. one prepared by Burkhardt, another by Schwarz, a third by Maser, Husserl, H. Müller, F. Rudio, C. Runge, a fourth by Haenlein and a fifth by Hettner; these copies relate to the lectures delivered in 1882, 1879 and 1875.

LIST OF WEIERSTRASS'S BERLIN LECTURES *

Year and Semester	Subject of the Lectures	Number of hours a week	Private, gratis or public
1856-57 Winter	Selected chapters of Mathematical Physics		Public
1857 Summer	General theorems relating to the representation of analytical functions by convergent series	1	"
	Theory of elliptic functions	4	Private
1857-58 Winter	Selected geometrical and mechanical problems soluble with the help of elliptic functions	2	Public
	Theory and application of trigonometric series and definite integrals which serve for the representation of arbitrary functions.	3	Private
1858 Summer	Some selected chapters of the Integral Calculus	2	Public
	Modern Geometry	4	Private

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* This list is taken from the list on pp. 355—360 of Vol. 3 of Weierstrass's *Mathematische Werke* and is based on the records of the University Treasury. It is not known for how many hours a week were the lectures of 1856—57 delivered.

Year and Semester	Subject of the Lectures	Number of hours a week	Private, or public
1858-59 Winter	General theorems relating to the representation of analytical functions by infinite series	1	Public
	Theory of elliptic functions	5	Private
1859 Summer	The homogeneous functions of the second degree in arbitrarily many variables	2	Public but not delivered
	A number of selected geometrical and mechanical problems treated with the help of elliptic functions	3	Private and gratis but not delivered
1859 60 Winter	The formulae of analytical Dioptrics	1	Public
	Indroduction into Analysis	5	Private
1860 Summer	Some selected chapters of Analytical Mechanics	2	Public
	Introduction into Analysis (continued)	4	private and gratis
1860-61 Winter	Theory of elliptic functions	6	Private
1861 Summer	Some selected problems of Geometry and Mechanics soluble with the help of elliptic functions	4	Public
1861-62 Winter	On surfaces of the second degree	2	Public but not delivered
	General theory of analytical functions	4	Private but not delivered

Year and Semester	Subject of the Lectures	Number of hours a week	Private, gratis or public
1862-63 Winter	Theory of elliptic functions	6	Private
1863 Summer	Selected problems of Geometry and Mechanics soluble with the help of elliptic functions	4	Public
	Introduction to the theory of Abelian functions	3	Private
1863-64 Winter	General theory of analytical functions	6	”
1864 Summer	Modern Synthetic Geometry	5	”
1864-65 Winter	Theory of elliptic functions	6	”
1865 Summer	On the applications of elliptic functions to geometrical and mechanical problems	3	Public
	Calculus of Variations	4	Private
1865-66 Winter	Theory of analytical functions	6	”
1866 Summer	Theory of Abelian transcendentals	4	”
	Modern Synthetic Geometry	4	”

Year and Sumester	Subject of the Lectures	Number of hours a week	Private, gratis or Public
1866-67 Winter	Theory of elliptic functions	6	Private
1867 Summer	Application of elliptic functions to geometrical and Mechanical Problems	4	Public
	Calculus of Variations	4	Private
1867-68 Winter	Theory of determinants and their applications	2	Public
	Modern Synthetic Geometry	4	Private
1868 Summer	Theory of analytical functions	6	„
1868-69 Winter	Theory of elliptic functions	6	„
1869 Summer	Theory of Abelian functions	4	„
	Different applications of elliptic functions	4	„
1869-70 Winter	Calculus of Variations	4	„
	Modern Synthetic Geometry	4	„
1868 Summer	Theory of analytical functions	6	„

Year and Semester	Subject of the Lectures	Number of hours a week	Private, gratis or Public
1868—69 Winter	Theory of elliptic functions	6	Private
1869 Summer	Theory of Abelian functions	4	„
	Different applications of elliptic functions	4	„
1869—70 Winter	Calculus of Variations	4	„
	Modern Synthetic Geometry	4	„
1870 Summer	Theory of analytical functions	6	„
1870—71 Winter	Theory of elliptic functions	8	„
1871 Summer	Modern Synthetic Geometry	4	„
	Selected geometrical and mechanical problems soluble with the help of the theory of elliptic functions	4	„
1871—72 Winter	Theory of Abelian functions	6	„
1872 Summer	Introduction to the theory of analytical functions	4	„
	Calculus of Variations	4	„

Year and Semester	Subject of the Lectures	Number of hours a week	Private, gratis or Public
1872—73 Winter	Theory of elliptic functions	6	Private
1873 Summer	Elements of modern Synthetic Geometry	4	,,
	Selected geometrical and mechanical problems soluble with the help of the theory of elliptic functions	4	,,
1873--74 Winter	Theory of Abelian funtions	6	,,
1874 Summer	Introduction to the theory of analytical functions	6	,,
1874—75 Winter	Theory of elliptic functions	6	,,
1875 Summer	Selected geometrical and mechanical problems soluble with the help of the theory of elliptic functions	4	,,
	Calculus of Variations	4	,,
1875—76 Winter	Theory of Abelian functions	6	,,
1876 Summer	Introduction to the theory of analytical functions	6	,,
	Supplements to the theory of Abelian functions	2	Private and gratis

Years and Semester	Subject of the Lectures	Number of hours a week	Private, gratis or public
1876—77 Winter	Theory of elliptic functions	6	Private
1877 Summer	Application of elliptic functions to the solution of geometrical and mechanical problems, illustrated by selected examples	4	„
	Calculus of Variations	4	„
1877—78 Winter	Theory of Abelian functions	6	„
1878 Summer	Introduction to the theory of analytical functions	6	„
	Application of Abelian functions to the solution of selected geometrical problems	2	Private and gratis
1878—79 Winter	Theory of elliptic functions	6	Private
1879 Summer	Calculus of Variations	4	„
	Application of elliptic functions to the solution of selected geometrical and mechanical problems	4	
1879—80 Winter	Theory of Abelian functions	6	„

Year and Semester	Subject of the Lecturss	Number of hours a week	Private, gratis or Public
1880 Summer	Introduction to the theory of analytical functions	6	Private (but not delivered)
1880—81 Winter	Introduction to the theory of analytical functions	5	Private
1881 Summer	Theory of elliptic functions	5	,,
1881—82 Winter	Theory of Abelian functions	5	,,
1882 Summer	Calculus of Variations	4	,,
1882—83 Winter	Introduction to the theory of analytical functions	5	,,
1883 Summer	Theory of elliptic functions	5	,,
1883—84 Winter	Theory of Abelian functions	5	Private (but undelivered)
1884 Summer	Calculus of Variations	5	Private
1884—85 Winter	Introduction to the theory of analytical functions	5	,,

Year and Semester	Subject of the Lectures	Number of hours a week	Private, gratis or public
1885 Summer	Theory of elliptic functions	5	Private
1885—86 Winter	Selected chapters of the theory of functions	4	Private (but) undelivered)
1886 Summer	Selected chapters of the theory of functions	5	Private
1886—87 Winter	Theory and application of bilinear and quadratic forms	3	”
1887 Summer	Theory of hyper—elliptic functions	5	”
1887—88 Winter	Calculus of Variations	$4\frac{1}{2}$	Private (but not delivered)
1888—89 Winter	Fundamental notions and principal theorems of the theory of functions	3	Private (but not delivered
1889—90 Winter	Calculus of Variations	3	Private

RIEMANN

〈 1826—1866 〉

CHAPTER VI.

RIEMANN

Georg Friedrich Bernhard Riemann was born on the 17th September, 1826, in Breselenz, a village in the Province of Hannover (then a kingdom under Great Britain), near Dannenberg and the Elbe. His father was Friedrich Bernhard Riemann, born in Boitzenburg on the Elbe in Mecklenburg, and his mother was Charlotte, daughter of Court Councillor Ebell of Hannover. There were six children born to the pair and Bernhard was the second of them. He had four sisters and a brother, Wilhelm, who became later Postal Secretary in Bremen.

Riemann's father, who had fought as a Lieutenant under Wallmoden in the wars against Napoleon, was at the time of Riemann's birth pastor at Breselenz; later he went with his family to the neighbouring pastorate of Quickborn where much of the early life of Riemann was spent and for which he ever retained the deepest affection. Very early, the desire to learn was stimulated in Riemann by his father who alone taught him almost up to the time of his admission into a Gymnasium. In the excellent biography of Riemann by his dear and intimate friend, Dedekind, we read :—"As a boy of 5 years he took much interest in History, in the battles of the ancient times and particularly in the unfortunate fate of the Poles.

Very soon, however, all these receded in the back ground and his decisive talents for Arithmetic broke path; he could not find greater pleasure than in framing difficult examples and setting them before his brother and sisters. Later, from the tenth year onwards, Bernhard's father allowed the teacher Schulz to help him in the instruction of his children; Schulz gave good instruction in Arithmetic and in Geometry but he had soon to exert himself very much to be able to follow the better and quicker solution of a problem by his pupil."

At the age of 13½ years Riemann left his father's house and came in Easter 1840 to Hannover where his maternal grandmother lived. Riemann was admitted into the third class of the Gymnasium and remained there a student for two years until the death of his grand-mother with whom he had resided. In the begining he had difficulties; but he overcame them and was praised for his quick progress in his various subjects of study. He was always an obedient and diligent student. But his thoughts were centred on his paternal house and whenever a vacation was near he would beg his parents most earnestly to let him spend it at Quickborn and would take great pains to find how at the least expense he would be able to complete the journey. His timidity in relations with strangers was very great and never completely left him as long as he lived. This timidity drove him in his later life to solitude and meditation in which he displayed the greatest boldness of thought and freedom from prejudice.

After his grand-mother's death, at the request of Riemann himself, his father got him admitted in Easter 1842 into the Gymnasium at Lüneburg where he spent two years in the second class and two years in the first class. Riemann found great difficulty in presenting punctually his school compositions in German and Latin. This was due to his habit of not coming quickly to a final decision in his writing: he would write a few lines, then, new thoughts coming to him, he would introduce them in the original draft, and later being dissatisfied would begin a new draft altogether. This defect was a source of great anxiety to the school-committee, and, as in other respects he was a good student, a way was suggested by the director of the school, Schmalfuss, at whose request the teacher of Hebrew, Seffer, took him as a paying guest in his own house so as to be able to look after the boy. Seffer took great pains with Riemann, sometimes sitting up to late in the night. Seffer found him one of his best pupils in Hebrew, which he studied as his father had destined him for a clergyman's life. In fact Riemann was of great assistance to Seffer when he was writing his *Elementary book of the Hebrew language.* According to Seffer who was also religious instructor to Riemann, during his days at Lüneburg, Riemann was very orthodox and this orthodoxy he retained all his life. Such was his faith in the Bible that when he was being looked upon as a great mathematician he was engaged in proving, according to Seffer, by starting on a mathematical basis, the truth of the Genesis and other biblical doctrines.

In a letter to Schering sent after Riemann's death, Schmalfuss reports on Riemann's mathematical studies at Lünebu-g ;—During his school days from the very beginning he had the reputation among his fellow students of being a good mathematician, When students were consulting one another as to the gifts to be sent to their parents on the occasion of a certain Christmas, Riemann said that he would present a perpetual calendar. He was laughed at by the boys at the time, but when he had the calendar ready not only were his little friends thunderstruck but also experts admired it. The calendar was not a copy but an original one based on Riemann's own thoughts. "I came to know soon the capacity of Riemann for understanding mathematical subjects." Schmalfuss placed at the disposal of Riemann all his mathematical books. He was exempted from attendance at mathematical lectures. "In the first year of his first class, he asked for the loan of a book and said in his modest manner 'it will be right pleasant for me if it is not too easy'. I pointed to him my book shelf and he picked up Legendre's theory of numbers, I said, 'Try what you can understand in it.' This happened on Friday afternoon. He brought back the book on the succeeding Thursday, 'How far have you read' ?, I said. He replied: 'that is surely a wonderful book. I know it thoroughly'. At the maturity examination, up to which time he had never had the book again before his eyes, he proved by working out the problems which I specially chose as examiner from Legendre's book that he knew it as if he had specially prepared himself for examination in it, The theory of numbers had a special attraction for him." "He read through Legendre's Geometry and worked out a large number of geometrical problems from books in my library......Already at that time he was such a mathematician that before his mathematical wealth the teacher felt himself to be poor." Schmalfuss points out a special peculiarity noticeable in Riemann at the time: 'he had an almost incredible gift of perception (Anschaung), of constructive fantasy and of at the same time most abstract generalization".

At the age of 19½ years, Riemann was admitted into the Goettingen University in Easter 1846 as a student of Philology and Theology in accordance with his father's wish. But, on account of his attraction towards Mathematics and specially his desire to ease the pecuniary circumstances of his father, whose means were limited and who had a large family, by getting a post earlier, Riemann soon gave up with his father's consent his preparation for a pastor's life. In the summer semester of 1846 he attended the lectures of Stern on the numerical solution of equations and those of Goldschmidt on Earth-magnetism; and in the winter semester of 1846—1847 he attended Gauss's lectures on the method of least squares and Stern's lectures on definite integrals. As Gauss's lectures were in those days restricted to a narrow field allied specially to applied Mathematics, Riemann's desire to enrich his knowledge by new ideas in Mathematics took him to the Berlin University in Easter 1847. The Berlin University had attracted at the time numerous mathematical students on account of the brilliance of the discoveries of the three Professors there, viz. Jacobi, Lejeune-Dirichlet and Steiner. Riemann remained at Berlin two years during which he attended the lectures of Jacobi on Analytical Mechanics and Higher Algebra, of Dirichlet on the theory of numbers, thetheory of definite integrals and the theory of partial differential-equations, Under Eisenstein he studied the theory of elliptic functions.

During his stay at Berlin, Riemann was much impressed by the events of the year 1848. He was an eye-witness of the March revolution and had been guarding the royal palace from 9 A. M. on the 24th March to 1 P. M. on the 25th as a member of the corps formed by the students. Before his return to Göttingen, he had been at Berlin at the time when the Frankfort Kaiser-Deputation arrived there.

After his return to Göttingen, Riemann was full of ideas and was also anxious to have his dissertation for the Ph. D. degree ready as early as possible. But he yielded to temptations with a view to earn money so as to help his father: he joined the newly founded mathematical physical Seminar conducted by Weber, Ulrich, Stern and Listing, and was appointed by Weber as an assistant in which capacity he helped in physical demonstrations and also delivered some lectures on Physics to beginners. These distractions were partly responsible for the delay in the presentation of his dissertation. But probably the chief reason was the painful anxiety which Riemann always showed in the writing out of papers meant for being printed.

In November, 1851, Riemann sent his dissertation "Foundations of a general theory of functions of a complex variable." The dissertation was well received by Gauss who informed Riemann during the course of an interview that he had for years been preparing a memoir on the same subject. The examination was fixed for the 3rd December and the open disputation and the conferment of the doctorate for the 16th December. At about this time he wrote to his father as follows; "By my present completed dissertation I hope to have considerably improved my prospects; also I hope to learn in course of time to write more fluently and rapidly, viz. when I go more into society and have opportunity to deliver lectures; for that reason I am now more cheerful". At the same time he asks his father to excuse him for not having tried more eagerly for the post of observer rendered vacant by the death of Goldschmidt and informs him that his habilitation will follow as soon as his thesis is ready.

The subject chosen in 1851 for his habilitationsschrift was known to be trigonometric series. In the autumn vacation he met Dirichlet at Goettingen where he was staying for a time and Riemann, who looked upon Dirichlet as the greatést living mathematician after Gauss, wrote to his father as follows: "The other morning Dirichlet was with me for two hours; he gave me the notes, which I require for my habilitationsschrift; they are so complete that they render the task of writing essentially easier; otherwise I would have had to seek in the library many things. Also he went with me through my dissertation and was altogether extremely friendly towards me, which on account of the great distance between his position and mine I would not have dared to expect." He was in those days writing to his father as if the presentation of his habilitationsschrift and the commencement of his lectures were things which would happen in the nearest future. But the habilitationsschrift was ready only in the beginning of December 1853. As regards the test-lecture, the custom was that out of the three subjects proposed by the candidate one was chosen. Riemann had prepared himself for the first two but it was the third which was chosen by Gauss. This subject was "Ueber die Hypothesen, welche die Geometrie zu Grunde legen." Riemann began to work out his test-lecture, and in summer, 1854, be came out with the two incomparable papers, "Uber die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe" and "Ueber die Hypothesen, welche die Geometrie zu Grunde legen" as his habilitationsschrift and habilitations-lecture.

What was thought at the time by experts about these three productions of Riemann may be briefly indicated here.

(I) Gauss's verdict* on the doctor-dissertation was the following:—

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* Schering's *Math. Werke*, Bd. 2, p. 375.

"The thesis submitted by Herr Riemann exhibits a convincing testimony to the thorough and deeply penetrating studies of the author in those regions which belong to the subject treated in the thesis, to a creative mind which is active and really mathematical and to a gloriously productive originality. The presentation is circumspect and concise and in parts elegant; the greatest part of the readers would have liked a greater transparency in the arrangement. The whole is a solid, valuable work not only coming to the standard which is demanded from theses for the doctorate but going far beyond it."

(II) Schering, who was first a pupil, and later a colleague, of Riemann, seems to have realized the importance of the Habilitationsschrift. He describes the original features of that work thus: "The new point of view here is an extension of the notion of the integral, so that he lays down, for example, the integral of such a function as is infinitely often discontinuous in every interval, ever so small, of its argument. As illustration, he gives, by means of very simple means, such a special function, nowhere considered up to that time, which has a finite saltus for every rational value of the argument (with even denominator). The function given by a Fourier series is investigated now not for its own properties but for the much simpler peculiarity of a function definitely connected with it, which will be equivalent to the second integral of the original function if that integral exists. The subject had for Riemann, for this reason great interest, because these series were used by Dirichlet with much success in proving theorems in the theory of numbers. However, the functions occuring in the theory of integral numbers make such high demands on the admissibility of discontinuities that one ought to concieve of functions which offer greater complications than those already used, functions which have infinite number of maxima and minima in finite intervals".*

(III) The test-lecture produced a great and immediate sensation. According to Dedekind, Riemann had taken special pains to make it as understandable as possible to the non-mathematical members of the Faculty of Philosophy. Gauss, who was present at the lecture, spoke† to Wilhelm Weber with the highest appreciation of Riemann's achievement.

We proceed now to consider in detail each of the three writings:—

(A)

The dissertation consists of 22 articles and the table of its contents, as given by Riemann‡ himself, is the following:—

1. A variable complex quantity $w=u+vi$ is called a function of another variable quantity $z=x+yi$,

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* Schering's *Werke*, Bd. 2, pp 378–379.

† See Dedekind's *Riemann's Lebenslauf* (Riemann's *Math. Werke*, p. 517).

‡ See Weber's edition of Riemann's *Mathematische Werke*, pp. 45—46.

when w so alters with z that $\frac{dw}{dz}$ is independent of dz. This definition is established by the remark that this independence always exists when the dependence of w on z is given by an analytical expression.

2. The values of the variable complex quantities z and w are represented by the points O and Q of two planes A and B, their dependence on one another as a representation of one plane on the other.

3. If the dependence is of such a kind that $\frac{dw}{dz}$ is independent of dz, then between the original and its image there is similarity in the smallest parts.

4. The condition, that $\frac{dw}{dz}$ is independent of dz, is identical with the following:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

From these follow

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

5. As the locus of the point O is substituted for the plane A a bounded surface T stretched over A. Turning points of this surface.

6. On the connectivity of a surface.

7. The integral $\int\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)dT$ extended

over the whole surface is equal to

$$-\int (X \cos \xi + Y \cos \eta)\, ds$$

over its whole boundary, if X and Y are arbitrary functions of x and y, continuous in all the points of T.

8. Introduction of the coordinates s and p of O with reference to an arbitrary line. The mutual dependence of the sign of ds and dp is so fixed that

$$\frac{\partial x}{\partial s} = \frac{\partial y}{\partial p}.$$

9. Application of the theorem in Art. 7 when in all the parts of the surface

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0.$$

10. Conditions under which in the inside of a surface T, simply covering A, a function u which, speaking generally, satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

is everywhere along with all its differential coefficients finite and continuous.

11. Properties of such a function.

12. Conditions under which in the inside of a simply-connected surface T simply covering A a function w of z is together with all its differential coefficients finite and continuous.

13. Discontinuities of such a function in an inner point.

14. Extension of the theorems of Arts. 12 and 13 to the points in the inside of an arbitrary plane surface.

15. General properties of the representation, of a surface T stretched in the plane A on a surface S stretched in the plane B, by means of which the values of a function w of z are geometrically represented.

16. The integral

$$\int\left[\left(\frac{\partial\alpha}{\partial x}-\frac{\partial\beta}{\partial y}\right)^2+\left(\frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial x}\right)^2\right]dT,$$

extended over the whole surface T, obtains, with the change of α by continuous functions or indeed functions discontinuous only in individual points, which are 0 on the boundary, always for one a minimum value and, if with the change in individual points removable discontinuities are excluded, only for one.

17. Establishment, by the method of limits, of a theorem assumed in the preceding article.

18. If in an arbitrarily connected plane surface T, resolved into a simply connected T* by cross-cuts, a function $\alpha+\beta i$ of x, y is given, for which

$$\int\left[\left(\frac{\partial\alpha}{\partial x}-\frac{\partial\beta}{\partial y}\right)^2+\left(\frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial x}\right)^2\right]dT$$

is finite through the whole surface, then that function can be always transformed into a function of z by the addition of a function $\mu+\nu i$ of x, y which is so conditioned: that (1) μ is zero on the boundary, ν is given in a point; (2) the alterations of μ in T, those of ν in T* are only in individual points discontinuous and only so that

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* *Entwicklungen*, l. c., p. 251.

$$\int\left[\left(\frac{\partial\mu}{\partial x}\right)^2+\left(\frac{\partial\mu}{\partial y}\right)^2\right]dT$$

and

$$\int\left[\left(\frac{\partial\nu}{\partial x}\right)^2+\left(\frac{\partial\nu}{\partial y}\right)^2\right]dT$$

remain throughout the whole surface finite and the latter is equal on each side of the cross-cut.

19. Calculation of the sufficient and necessary conditions for the determination of a function of a complex variable inside a given domain.

20 The earlier method of determining a function by means of operations on quantities contains superfluous parts. By the considerations made here the extent of the determination pieces of a function is referred back to the necessary standard.

21. Two given simply connected surfaces can be always so referred to one another that to each point of the one corresponds one point of the other, changing continuously with the first point, and the smallest parts of the surfaces are similar; and indeed to an inner point and to a boundary point the corresponding can be arbitrarily given. Thereby is the relation determined for all points.

22. Concluding remarks.

The following criticisms of the dissertation, which, according to Klein*, produced apparently no immediate impression, are of importance :—

(I) The definition of a function of z is the same as that given much earlier by Cauchy.

(II) The two partial differential equations for u and v were given much earlier by Cauchy.

(III) The notion of conformal representation originated much earlier with Gauss.

(IV) "We should not at all require in Riemann's works the same logical rigour as in Weierstrass's works. However, by the richness of his ideas and the multitude of his points of view, Riemann succeeds always in getting at the essentials".†

(V) The introduction of Riemann's surfaces into Analysis is a real achievement of Riemann which has proved fruitful.

(VI) Weierstrass ‡ says about the definitions of Cauchy and Riemann :—

(1) the domain for which the function is defined is not clear ; (2) nothing is said in Cauchy's first definition how the function can be continued beyond its domain ; (3) this defect is partially remedied in the case of the second definition by Riemann who says that if two functions u_1, v_1 exist, which in a domain common to them and u, v satisfy the two partial

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* l, c., p.251.

† Klein's *Entwicklungen*, l. c., p. 256.

‡ "Zur Funktionentheorie" (from a lecture delivered in 1884), *Acta Math*, t. 45, pp. 1–10 specially pp. 9—10

differential equations and are in that part identical with u and v respectively, then u_1+iv_1 may be called the continuation of $u+iv$. The *proof* of the possibility of such a continuation is not to be found in any publication of Riemann: in his lectures he gave a proof based on the expansibility of $u+iv$ in powers of $z-z_0$. Both for determining the domain of the function and for its continuation the resort to power-series in $z-z_0$ is found necessary by Riemann; and that involves not only complications but numerous uncalled for assumptions.* Weierstrass thus proves the naturalness and simplicity of his own definition of $f(z)$ as the sum of a power-series.

(B)

The Habilitations-schrift begins with the following introduction which is followed by 13 articles:—

"The following paper on trigonometric series consists of two essentially different parts. The first part contains a history of the investigations and views on the arbitrary (graphically given) functions and their representability by means of trigonometric series. In the course of its compilation it was my good fortune to use some hints of the famous mathematician to whom one owes the first well-grounded work on this subject. In the second part I furnish, on the representability of a function by means of trigonometric series, an investigation which also comprehends the hitherto unsettled cases. It was necessary to preface it by a brief essay on the notion of a definite integral and the range of its validity."

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* e. g. the assumption of the existence of the differential coefficient of every continuous function which is implicitly made in the procedure of Riemann.

The following is the table of the contents of the 13 articles:—

History of the question of the representability of a function by means of a trigonometric series.

§ 1. From Euler to Fourier.

Origin of the question in the controversy over the importance of d' Alembert's solution and Bernoulli's solution of the problem of the vibrating strings in the year 1753. Opinions of Euler, d' Alembert, Lagrange.

§ 2. From Fourier to Dirichlet.

The correct opinion of Fourier disputed by Lagrange (1807). Cauchy (1826).

§ 3. Since Dirichlet.

Settlement of the question by Dirichlet for the functions met with in Nature. Dirksen (1829). Bessel (1839).

On the notion of a definite integral and the range of its validity.

§ 4. Definition of a definite integral.

§ 5. Conditions of the possibility of a definite integral.

§ 6. Particular cases.

Investigation of the representability of a function by means of a trigonometric series without particular suppositions on the nature of the function.

§ 7. Plan of the investigation.

I. On the representability of a function by means of a trigonometric series whose coefficients ultimately become infinitely small.

§ 8. Proofs of some theorems important for the investigation.

§ 9. Conditions for the representability of a function by means of a trigonometric series with coefficients diminishing indefinitely.

§ 10. The coefficients of the Fourier series become ultimately infinitely small, if the function to be represented remains throughout finite and admits of an integration.

II. On the representability of a function by means of a trigonometric series with coefficients not diminishing indefinitely.

§ 11. Referring back of this case to the preceding case. Consideration of particular cases.

§ 12. Functions which do not have an infinite number of maxima and minima.

§ 13. Functions which have an infinite number of maxima and minima.

As regards the influence of this work, it is no exaggeration to say that by his new notion of a definite integral Riemann enriched Analysis by indirectly introducing changes in the minds of mathematicians regarding the fundamental properties of a function of a real variable, e. g. about the possibility of the points of discontinuity or the points of non-differentiability being everywhere dense. The notion itself marked a great advance over the notions of Cauchy and Dirichlet—an advance which had been vainly attempted by mathematicians before Riemann. According to Hobson,[*] "the definition given by Riemann leaves nothing to be desired as regards precision. Riemann not only formulated a rigorous arithmetical definition of the integral of a bounded function, but also established a necessary and sufficient condition for the existence of the definite integral of the function. This definition of the definite integral was in the latter part of the 19th century, the one that was employed in all rigorous Mathematical Analysis."

About the part of the Habilitations-schrift dealing with the convergence of trigonometric series, A. Harnack[†] says: Riemann's proof in Art. 9 of his memoir has remained altogether unalterably the foundation for the theory of trigonometric series, and his theorem in Art. 13 showed the way for finding the limits of representability which Herr Du Bois—Reymond for the first time determined for continuous functions at definite points."

(C)

The Habilitations-Vortrag or Test-Lecture begins with the following :—

Plan of the Investigation.

"It is known that Geometry assumes as given the notion of space as also the first fundamental notions for the constructions in space; Geometry gives only nominal definitions of them whilst the essential determinations occur in the form of axioms. The relation of these suppositions remains thereby in the dark; one sees neither, if and to what extent their connection is necessary, nor a priori whether such connection is possible.

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[*] *Theory of functions of a real variable*, Vol. 1, 3rd edition, p. 459.

[†] *Math. Ann.*, Bd. 19, 1882, p. 236.

"This darkness was not removed from the time of Euclid up to Legendre, to mention only the most famous writers of Geometry, either by the mathematicians or by the philosophers who engaged themselves with the subject. The reason for this was that the general notion of multiply-extended quantities, amongst which the space quantities are contained, remained completely neglected. I have therefore first placed before myself the problem to construct the notion of a multiply-connected quantity from general quantity-notions. It will follow from that that a multiply-extended quantity is capable of different measure-relations and the space forms therefore a particular case of a triply-extended quantity. From this there is, however, a necessary consequence, that the theorems of Geometry do not admit of derivation from general quantities-notions but that those properties by means of which the space is distinguished from other conceivable triply-extended quantities can be taken only from observations. Out of this arises the problem, to seek the simplest facts from which the measure-relations of space admit of determination—a problem which, according to the nature of the thing is not fully definite; for, several systems of simple facts admit of being given which suffice for the definition of the measure-relations of space; the most important for the present aim is the one laid at the foundation by Euclid. These facts are like all facts not necessary but only of empirical certainty, they are hypotheses; one can, therefore, investigate their probability, which is surely very great inside the limits of observation, and thereafter judge the admissibility of their extension on each side of the limits of observation, on the side of the immeasurably large as well as on the side of the immeasurably small.*

The outline of the memoir is as follows:—

Plan of the investigation.

I. Notion of an *n*-ply extended quantity.

§ 1. Continuous and discrete varieties. Definite parts of a variety are called quanta. Division of the study of continuous quantities into the study

(1) of the pure domain-relations in which an independence, of the quantities, of the place is not assumed,

(2) of the measure-relations in which such an independence must be assumed.

§ 2. Generation of the notion of a simply, doubly,... *n*-ply extended variety.

§ 3. Referring back of the place-determination in a given variety to quantity-determinations. Essential criterion of an *n*-ply extended variety.

II. Mass relations of which a variety of *n*-

dimensions is* capable, under the supposition that the lines possess a length independent of the position and therefore every line is a measurable line by every other line.

§ 1. Expression of the line element. As plane are considered such varieties in which the line element is expressible by the root of the sum of the squares of complete differentials.

§ 2. Investigation of n-ply extended varieties, in which the line-element can be represented by the square root of a differential expression of the second degree. Measure of its deviation from plane-ness (measure of curvature) in a given point and a given direction of the surface. For the determination of the mass-relations it is (under certain restriction) admissible and sufficient that the measure of curvature in every point be given arbitrarily in $\frac{n(n-1)}{2}$ directions of the surface.

§ 3. Geometrical illustration.

§ 4. The plane varieties (in which the measure of curvature is altogether $=0$) admit of being considered as a particular case of the varieties with constant measure of curvature. These can be also defined by this, that in them there is independence, of the n-ply extended quantities, of the position (moveability of the same without extension).

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* The investigation on the possible measure-determinations of an n-ply extended variety is very incomplete, although fairly sufficient for the present object.

§ 5. Surfaces with constant curvature.

III. Application to the space.

§ 1. Systems of facts, which suffice for the determination of the measure-relations of the space as Geometry assumes them.

§ 2. How far is the validity of these empirical determinations probable within the limits of the observation in unmeasurable quantities ?

§ 3. How far in infinitely small quantities? Connection of this question with explanation of Nature.*

To the considerations relating to non-euclidean Geometry which his contemporaries, Gauss, Lobatschefsky and J. Bolyai, had made familiar to mathematicians, Riemann, by his test-lecture, gave "a new and specific turn in this way, that he placed in the forefront the ideas of *analytical* Geometry: the space appears to him as a particular case of a triply-extended number-variety, in which the square of the arc-element admits of being expressed by a quadratic form of the differentials of the coordinates .... The essential in this connection is that Riemann here also has remained true to his fundamental thoughts: *to understand the properties of things from their behaviour in infinitely small.* He has thereby laid the foundation of a new chapter of Differential Calculus: *about the theory of the quadratic differential expressions of arbitrary variables* and about the theory of the *invariants* which these differential expressions possess with respect to arbitrary transformations of the variables".* The importance of Riemann's test-lecture may be gauged from the fact that, according to some eminent mathematicians, including Klein†, Einstein's theory of relativity is simply the theory of invariants of the four-dimensional space-time-region x, y, z, t with respect to a definite group of collineations.

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* § 3 of III requires still an overhauling up and further construction.

H. Weyl thus summarises the foundations of the Riemannian Geometry, an n-dimensional Riemannian space being an n-dimensional variety, not, however, an arbitrary one but one in which the measure is taken by the stamp of a definite quadratic differential form:—

"The two fundamental laws according to which that form fixes the measure-quantities are the following (the numbers x_i denoting any coordinates):—

(1) If g is the determinant of the coefficients of the fundamental form, then the magnitude of any space-piece is given by the integral

$$\int \sqrt{g}\, dx_1\, dx_2 \ldots dx_n$$

which is to be taken over that mathematical domain of the variables x_i which corresponds to the space-piece.

(2) If $Q(d\delta)$ denotes the symmetrical bilinear form, corresponding to the fundamental form, of two line-elements d and δ at the same place, then the angle θ formed by them is to be calculated from

$$\cos\theta = \frac{Q(d\delta)}{\sqrt{(Q(dd).\ Q(\delta, \delta))}}."\ \ddagger$$

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* Klein, *Gesammelte Math. Abhandlungen*, Bd. 3, p. 495.

† l. c., Bd. 1, p. 539.

‡ *Raum, Zeit, Materie*, 4th edition, 1920, p. 84

Among the other important papers of Riemann may be mentioned :

I. "Beiträge zur Theorie der durch die Gauss'sche Reihe F(α, β, γ, x) darstellbaren Functionen" (*Memoirs of the Academy of Sciences of Gottingen*, 1857; *Werke**, pp. 67—83.)

II. "Theorie der Abel'schen Functionen" (*C. J.*, Bd. 54, 1857 ; *Werke*, pp. 88—142).

III. "Ueber die Anzahl der Primzahlen unter eine gegebenem Grösse" (*Monatsberichte Berliner Akademie*, 1859 ; *Werke*, pp. 145—153).

IV. "Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite" (*Memoirs*, 1860 ; *Werke*, pp. 156—175).

V. "Ein Beitrag zu den Untersuchungen über die Bewegung eines flüssigen gleichartigen Ellipsoids" (*Memoirs*, 1861 ; *Werke*, pp. 182—211).

VI. "Ueber das Verschwinden der Theta—Functionen" (*Crelle's Journal*, Bd. 65, 1865 ; *Werke*, pp. 212—224).

VII. "Ein Beitrag zur Electrodynamik" (*Poggendorf's Annalen*, 1858 ; *Werke*, pp. 288—294)

VIII. "Ueber die Fläche vom kleinsten Inhalt bei gegebener Begrenzung" (*Memoirs*, Bd. 13 ; *Werke*, pp. 301—334).

IX. "Zwei allgemeine Sätze über lineare Differentialgleichungen mit algebraischen Coefficienten" (*Werke*, pp. 379—390).

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* The edition of 1892 is meant throughout this list.

X. "Commentatio mathematica, qua respondere tentatur quaestioni ab Ill^{me} Academie Parisiensi propositae", 1861. (*Werke* pp. 391–402). Anmerkungen von Weber (*Werke*, pp. 405—423).

XI. "Ueber das Potential eines Ringes" (*Werke*, pp. 431—436).

XII "Zur Theorie der Abelschen Functionen" (*Werke*, pp, 487—504).

In summer 1854, Riemann became Privat-Dozent at Göttingen and in the succeeding winter-semester he delivered his first course of lectures, the subject being the theory of the partial differential equations with applications to physical problems. As a model for his lectures, in fact, he used the lectures which Dirichlet had delivered at Berlin on the same subject. At his first lecture there had been present eight persons, which was in excess of his expectations. Writing to his father on the 18th November, 1854, Riemann says : "My life here has now by and by taken a rather regular and uniform shape. I have been able to hold my lectures up to now regularly, my shyness of the beginning has already rather subsided, and I habituate myself to think more of the audience than of myself and to read from their miens whether I should proceed further or explain the matter." Dedekind says that there is no doubt that in the early years of the career of Riemann as a teacher the lectures which he had to deliver were a source of great difficulty to him. His brilliant capacity for thought and his phantasy which enabled him to guess things made him take long steps, specially in oral intercourse which one could not easily follow, and when he was asked to explain some of the intermediate links he was apt to become confused; it caused him some trouble to fit himself into the slower course of thought of others and to remove quickly their doubts. "Thus also has the observance of the miens of his audience, of which he writes above, often disturbed him sensibly, when he at times quite against his expectation believed it necessary to prove particularly a point which he had considered to be almost self evident. This disappeared after longer practice, and the comparatively larger number of his pupils is to be attributed not only to the attraction of his name, grown famous by the most deep-meaning works, but also to his lectures, which he prepared very carefully and by means of which it was possible for him to remove from the path of his audience those obstacles which stood in the way of their penetrating into the new principles created by him."

After the death of Gauss on the 23rd February, 1855, Dirichlet came soon from Berlin as his successor; and this opportunity was availed of by Riemann's friends to try to secure his appointment as Assistant Professor but without success: all that they could obtain for him was a remuneration of 200 thalers annually from the Government. This, however, gave much relief to Riemann who had begun to look into his future with gloomy eyes. About this time he suffered a number of calamities, one after the other: his father and his sister Clara died in 1855, the old beloved house in Quickborn was forsaken and his three sisters went to live at Bremen with his brother, Wilhelm, who was secretary to the Post office there.

With the appointment of his friend Dirichlet at Göttingen, it became possible for Riemann to choose himself the subject of his lectures and for the first time the theory of Abelian functions was lectured upon at Göttingen from Michaelmas 1855 to Michaelmas 1856; these lectures were attended by three persons, viz. Schering, Bjerknes, and Riemann's colleague, Dedekind. In summer 1856, he was appointed to be an assessor of the mathematical class of the Royal Society of sciences of Göttingen and on the 2nd November he communicated his memoir on Gauss's series. On the same day he wrote to his brother as follows: "Also I hope that my works will bear fruit. My memoir, as I wrote to you before, is ready for the press and, perhaps the Society will publish it in its *Memoirs*, which will be indeed a great honour since the *Memoirs* of the Society in the last 50 years have contained only the mathematical memoirs of Gauss, The mathematical section of the Society, consisting of Weber, Uirich and Dirichlet, will at least after Weber's expresssions of his opinion decide on the printing of my memoir. With my lectures, *i. e.* with the attendence at them, I am rather satisfied, specially in view of the small number of newly admitted students. Among them there are no mathematicians and that is also the reason why Dedekind and Westphal have not been able to settle about their private lectures. The number of my audience on the four days that I have lectured was three on the first day, then four, and five in the last two days; although among them was a guest. Very pleasing will it be to me if I were to have this time some of the audience from the first semester and not as usual purely from the 6th and later semesters, as this will be considered by me as a sign that my lectures have become more easily understandable. However, I cannot as yet assert that my lectures have come to stay; for, as yet no one has formally reported himself to me and it is always possible that my audience may leave me in the lurch. From now onwards I will spend all my free time on my work on the theory of Abelian functions of which I have already told you. Shortly before my return, the chief editor of the mathematical journal, Dr. Borchardt of Berlin, has been here in Göttingen, and invited me through Dirichlet and Dedekind to send him as soon as possible a description of my investigations on the theory of Abelian functions, however rough it may be. Weierstrass is now publishing rapidly, yet, as Scherk told me, the part published now contains only the first preliminaries to his theory".

Riemann devoted himself now with all his powers to the preparation of his theory so that he sent to Berlin the manuscripts of the first three small memoirs on the 18th May, 1857, and the manuscipt of the 4th larger one on the 2nd July. But on account of the great exertion, his health broke down about the end of the summer semester and for recuperating it he went to Harzburg where he stayed for some weeks, taking many walks and making long excursions in the Harz. Soon after his return to Göttingen, he was appointed Assistant Professor on the 9th November, 1857, and his remuneration was raised to 300 thalers from 200 thalers. But about this time he received the greatest shock due to the death of his dearly beloved brother, Wilhelm. This made him insist on his three sisters coming to stay with him but two of his sisters came only in the beginning of March, 1853, after the death of the youngest sister, Marie.

On the 5th of May, 1859, occurred the death of Dirichlet who had from the very beginning of their acquaintance felt the most lively interest in Riemann and had done his best to improve Riemann's external relations. In the mean while the scientific importance of Riemann was so widely recognized, that on the death of Dirichlet, the Government did not think of bringing an outsider to succeed him: Riemann was given a residence in the observatory in Easter, 1859, was appointed ordinary professor on the 30th July and was elected ordinary member of the Society of sciences in December. Already in August he had been elected a corresponding member of the Berlin Academy of sciences and had visited, in September, Berlin where he was received with distinction by Kummer, Borchardt, Kronecker and Weierstrass. In the Easter vacation of 1860 he made a journey to Paris where he lived for a month from the 26th March onwards and was very well satisfied with the reception he got at the hands of the learned world of Paris, including Serret, Bertrand, Hermite, Puiseaux and Briot. In 1861 he sent in his solution of the Prize-problem of the Paris Academy on the theory of conduction of heat, which, however, did not secure the prize because of the necessary calculations being incomplete.

Many honours were conferred on Riemann during the remaining period of his life. He had been elected by the Bavarian Academy of Sciences to be a corresponding member on the 28th November, 1859 and was elected an ordinary member on the 28th November, 1863. The Berlin Academy made him a foreign member in March, 1866 and in the same month he was made a corresponding member of the Academy of Sciences of Paris. Shortly before his death, on the 14th June, 1866 he was made a foreign member by the Royal Society of London. On the 3rd June, 1862, Riemann was married to Fräulein Elise Koch of Körchow in Mecklenburg-Schwerin, a friend of his sisters. But soon after that event, in July he had an inflammation of the skin of the breast which appeared to heal quickly but left the seeds of a lung disease which ultimately brought about his death.

At the urgent request of Wilhelm Weber and Sartorius von Walterhausen, Riemann was granted the necessary leave and also a substantial subsidy in order that he might stay for a long time in a milder climate as advised by the physicians. The journey to Italy, the first of the three that he was destined to undertake before his death, was begun in November, 1862. His stay of several months near Messina improved his health quickly and he started on the return journey on the 19th March, 1863, but shortly after his arrival in Göttingen on the 17th June his health became so bad on account of his having caught cold in the way that he had to start on a second journey to Italy on the 21st August. He arrived at Pisa after passing Meran, Venice and Florence. At Pisa on the 22nd December, 1863, his daughter Ida was born. In May, 1864 he lived in a villa near Pisa but unfortunately here he lost his youngest living sister, Helene. It was at about this time that he was offered the chair at Pisa University left vacant by Mossoti, but after careful consideration he declined it. In September his condition became so bad at Pegli near Genoa that he decided to return to Göttingen.

On his arrival at Göttingen on the 2nd October, Riemann lived through the winter with fairly good health so that every day he was able to work for some hours. Here he completed the memoir on the vanishing of the theta—functions and charged his former pupil, Hattendorf, with the editing of the memoir on minimal surfaces. It was his frequently expressed wish to consult Dedekind about some unfinished papers, but on account of great weakness he did not like to trouble Dedekind by asking him to come to Göttingen. In the last months of his stay at Göttingen Riemann was engaged in the editing of his memoir on the mechanic of the ear which unfortunately he left unfinished. He wished at this time to see Lago Maggiore again and stay there for some months so that with improved health he might complete the memoir on the mechanic of the ear and so he started on his third and last journey to Italy on the 15th June, 1865. He arrived at Lago Maggiore on the 28th June. His strength diminished rapidly and he saw fully well that his end was near. But up to a few days before his death he continued to work on his last paper, lying peacefully under a fig tree and fully enjoying the beautiful landscape. He died on the 20th July, 1866, at Selasca, in great peace, and was buried there.

I proceed now to give a brief account of each of the important papers I-XII mentioned on pp. 315—316:—

(I)

Before Riemann the hypergeometric function had been studied by Gauss and Kummer, with the differential equation, satisfied by the function, as the starting point. Riemann adopted an entirely new method by the introduction of the complex variable on the basis of the principles laid down by him in his doctor—dissertation; according to which principles, a function is defined by its boundary conditions and its discontinuties and all the other properties are derived from these; in particular are derived in this manner the formulae and representations for the function.

Riemann introduces his P—function

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

as a function of x which satisfies the following conditions :—

(1) It is single-valued and finite for all values of x with the exception of a, b, c.

(2) Among any three of these functions, say P_1, P_2, P_3, there holds a linear homogeneous equation with constant coefficients, viz.

$$C_1 P_1 + C_2 P_2 + C_3 P_3 = 0.$$

(3) The function admits of being put in forms $c_\alpha P^{(\alpha)} + c_{\alpha'} P^{(\alpha')}$, $c_\beta P^{(\beta)} + c_{\beta'} P^{(\beta')}$, $c_\gamma P^{(\gamma)} + c_{\gamma'} P^{(\gamma')}$ with constant coefficients c's so that

$$P^{(\alpha)} (x-a)^{-\alpha}, \quad P^{(\alpha')} (x-a)^{-\alpha'}$$

remain single-valued for $x=a$ and are neither infinite nor zero. Similarly for

$$c_\beta P^{(\beta)} + c_{\beta'} P^{(\beta')} \text{ and } c_\gamma P^{(\gamma)} + c_{\gamma'} P^{(\gamma')}$$

with respect to b and c respectively. For the six quantities

$\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, it is assumed that none of the differences

$$\alpha-\alpha', \beta-\beta', \gamma-\gamma'$$

is an integer; also it is supposed that

$$\alpha+\beta+\gamma+\alpha'+\beta'+\gamma'=1.$$

From the definition of P, a number of its properties are derived immediately, including the property that all P's with the same $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ can be referred to

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix},$$

which for brevity is denoted by $P\begin{pmatrix} \alpha, & \beta, & \gamma & x \\ \alpha', & \beta', & \gamma' & \end{pmatrix}$. In this manner, Riemann obtains quickly the relations between allied functions and specially the differential equations from which the representations of the hypergeometric function in power series and definite integrals are deduced. It should be noted that one of the relations following immediately from the definition is

$$P\begin{pmatrix} \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{pmatrix} = x^{\alpha}(1-x)^{\gamma} P\begin{pmatrix} 0 & \beta+\alpha+\gamma & 0 & x \\ \alpha'-\alpha & \beta+\alpha+\gamma & \gamma'-\gamma & \end{pmatrix}$$

and this is equal to

$$\text{constant} \times x^{\alpha}(1-x)^{\gamma} \quad F\,(\beta+\alpha+\gamma,\ \beta'+\alpha+\gamma,$$

$\alpha-\alpha'+1, x)$. About this paper and (II), Klein[*] says: "They have worked like revelations and have aroused the unhesitating wonder of experts".

(II)

This paper contains, what may be called an introduction of four articles, followed by 27 articles divided into two parts The articles of the introduction bear the headings: General suppositions and auxiliary means for the investigation of functions of unrestrictedly variable quantities.; Theorems of analysis situs for the theory of the integrals of complete differentials with two members; Determination of a function of a complex variable by means of the conditions of boundary and discontinuities; Theory of Abelian functions. This last article contains an outline of the two parts of the theory that follow.

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[*] *Entwicklungen*, . c., p. 252.

Riemann says that with certain exceptions the contents of the paper are taken from the lectures delivered by him from Michaelmas 1855 to Michaelmas 1856, and that the various results were discovered between 1851 and 1856.

Weierstrass* says: "A direct solution of my problem I have already presented to the Berlin Academy. The manuscript, which had been sent to the press, was, however, taken back by me; because a few weeks later Riemann published a paper, on the same problem, which rested on quite another basis than my paper and it did not admit of being recognized without difficulty that Riemann's results completely agreed with mine".

(III)

Klein† says about this paper: This famous work has become "the foundation for numerous works of the most recent times... The basis of this work is the so-called ζ— function on the zeros of which Riemann made definite pronouncements all of which have not been proved as yet, inspite of the most zealous effort by most various mathematicians".

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* *Acta Math.*, Vol. 39, p. 54.

† *Entwicklungen*, l. c., p. 252.

(IV)

This paper, which shows the influence of Wilhelm Weber on Riemann's scientific activity, is thus noticed by Riemann himself: This investigation makes no claim of furnishing useful results to the experimental researcher; the author wishes it to be considered only as a contribution to the theory of non-linear partial differential equations ... Indeed the solution of quite special problems which is the subject of this paper demands new methods and conceptions and leads to results which will probably play a part also in the solution of more general problems."

... ...

"The paper deals with the motion of the air or a gas only for the case when initially and also subsequently the motion is in every direction the same." The paper has been a source of inspiration to many writers, of whom Christoffel * may be mentioned.

(V)

The paper "is intended to carry on the further development of the beautiful thought by which Dirichlet crowned his scientific activity"; the reference in the above words of Riemann being to Dirichlet's paper† on the motion of a uniform fluid ellipsoid subject to the attraction of its elements according to Newton's law. Riemann uses Lagrange's dynamical equations referred to moving axes and finds a system of ten

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* *Gött. Nach.*, 1859; *Werke*, pp, 176—178.

† *Gött. Abh.*, *1860; Werke*, Bd. 2, pp. 265—301

equations which have three integrals giving the constancy of energy, of the vortex-strength and of the motion. The various kinds of stationary motion are considered and it is proved that the forms of Jacobi and Dedekind are stable, and that the form of Maclaurin is stable only when the ratio of the least semi-axis to the greatest >. 303327.

(VI)

In the second part of the memoir outlined under (II), Riemann has considered the vanishing of theta-functions; the object of the present paper is to rectify certain defects in the treatment of the subject in the afore-mentioned part.

(VII)

This paper was presented to the Royal Society of Sciences of Göttingen in 1858 and was later withdrawn on account of an error which Riemann noticed in it.

(VIII)

This paper was published first in the year 1867 in the *Memoirs of the Royal Society of Sciences of Göttingen* and overlaps to a considerable extent the previously published papers of Weierstrass and his pupil, Schwarz, to whom the Berlin Academy awarded its prize for the memoir "Determination of special minimal surfaces."

(IX)

This paper, although bearing the date 20. 2. 1857, came to light first in 1876 in the first edition of Riemann's *Werke*; and produced a great sensation at the time as in it are given some results which Fuchs and his school produced after 1864. The problem treated in it, though incompletely, is the following* :—"If one prescribes monodromy group and singular points $a_1, a_2, \ldots a_n$ of a differential equation of Fuchsian class, then does there correspond always a differential equation which possesses these singular points and this monodromy group ?"

(X)

The paper was sent to the Paris Academy on the 1st of July, 1861, as containing the solution of the following prize-problem announced in 1858 :—

"To find what ought to be the calorific state of an indefinite homogeneous solid for which a system of curves, isothermal at a given instant, remain isothermal at all subsequent times, in such a way that the temperature of a point can be expressed as a function of time and two other independent variables". As stated already on page 320 of this volume, the prize was not awarded to Riemann ; on account of the bad condition of his health he had failed to give complete proofs of his results. The paper appears in his *Werke* with copious notes by Weber. It may be noted that the problem was withdrawn in 1868 and probably no further attempts have been made to solve it.

(XI)

This paper is of interest because in it appear the functions, now called toroidal functions or ring-functions. Neumann gave them in a paper in 1864, very probably after the date of Riemann's paper.

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* See Bieberbach's *Differentialgleichungen*, 2nd edition, p 212.

(XII)

This paper deals with the developments of the theory of charateristics, which Riemann gave in his lectures of the winter semester of 1861—62 and which were first published after his death in his *Werke*.

The special importance of Riemann's courses of lectures, a list of which is given at the end of this chapter, came to be recognized very late, in fact as late as 1897, when copies of some of them came to light*. About these lectures, Klein† says: "They open to us a glimpse of the comprehensive insight of Riemann into things, which people believed to have discovered years after his death but which he had already given out in his lectures. Here one sees again how the development of Science depends on chance. How differently would have Mathematics progressed if either the audience of Riemann had had a deeper understanding for his course of thought than they actually had or we had found earlier the copies of his lectures. And how much material must have been lost for want of understanding and proper estimate of its value." The three courses of lectures, already published, are briefly described below:—

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* See *Gött Nachrichten*, 1897.

† Klein's *Entwicklungen*, l. c, p 247.

(I)

Vorlesungen über die partielle Differentialgleichungen und deren Anwendung auf physicalische Fragen. Edited by K. Hattendorf, Brunswick, 1869.

These lectures are based on (1) Riemann's copy of the first course of lectures delivered by him at the University in 1854, (2) the copy, prepared by Hattendorf, of the course of 1860–61 and (3) the additions made by Riemann in the course of 1862. The contents are:—The theory of definite integrals, that of Fourier series, that of partial differential equations; the applications to the theory of heat, namely to the conduction of heat in an infinite body and a sphere, whose surface is uniformly heated; the theory of elastic strings and of air waves; the general case of the conduction of heat in a sphere; the equations for the elastic rigid bodies with examples; and the motion of a rigid body in a finite liquid.

The lectures do not contain any original work of Riemann himself. They have been brought out by H. Weber in entirely extended forms from time to time until we have now two volumes of very interesting matter in the book, of over 1500 pages, *Die Differential-und Integralgleichungen der Mechanik und Physik* (Brunswick, 1925 and 1927).

(II)

Schwere, Elektricität und Magnetismus. After the lectures of Bernhard Riemann, edited by K. Hattendorf, Hanover, 1876.

The book consists of 358 pages divided into two parts and altogether 111 articles. The first part is headed: "Gravitation. General theorems on the potential-function and the potential" and is divided into three sections. The second part is headed: "Electricity and magnetism" and is divided into six sections. The headings of the nine sections are the following:—The potential-function (pp. 3—68); the theorem of Green (pp. 69—151); anxiliary theorems from Mechanics (pp. 152—178); Electrostatics (pp. 179—214); Galvanic currents (pp. 215—242); Magnetism, Electromagnetism and Electrostatics (pp. 243—305); Induction (pp. 306—312); the fundamental rule of electrical mutual action (pp. 313—337); terrestrial magnetism (pp. 338—358).

The following lines from the preface are of interest: "This book has originated from the lectures which Riemann delivered on gravitation, electricity and magnetism in the summer semester of 1861 at Göttingen. Apart from some quite brief notes, there exists no manuscript by Riemann himself about these lectures." For the exposition I alone am responsible......As in the case of the partial differential equations so in this case also we have to remember Lejeune Dirichlet. Together with his great services for the advancement of Science, it should not be forgotten that he has been the first to have lectured at German Universities on partial differential equations and on the potential. Such lectures have not ceased on account of his death: they form at nearly all Gernan Universities a regular part of the programme. Thus also Riemann took up the lectures after Dirichlet. With the identity of the subject, it is natural that in the plan and the execution much is the same as with Dirichlet. But Riemann has not restricted himself simply to the inheritance from his great predecessor. The expert will find that Riemann has given a multitude of what is peculiar to him".

(III))

Vorlesungen über elliptischen Functionen. Edited with additions by Hermann Stahl. Leipsic, 1899.

The book of 144 pages is divided into two parts; the first part bearing the heading "Riemann's theory of elliptic functions", and the second part the heading "Explanations and supplements". The first part is divided into five sections I—V which are respectively headed as follows: The doubly-periodic functions; the integral of the first kind and the elliptic functions; analytical representations of the elliptic functions and integrals; addition, transformation and multiplication of the elliptic functions; theory of the elliptic functions starting from the theta-functions. The second part contains five sections which bear respectively the headings:— Preparatory theorems for section II; additions to section II; additions to section III; additions to section IV; the theory of Weierstrass.

In the preface, Stahl says: "The *Vorlesungen über elliptischen Functionen*, published here, form a part of the lectures, which Riemann delivered twice in different forms (under the title: On functions of a complex variable, in particular elliptic and Abelian), first in Winter 1855–56 and Summer 1856 and then in Winter 1861–62 and Summer 1862. In both the courses the section on elliptic functions forms an independent whole, even if it appears first as an example of the theory of Abelian functions".

The first four sections, viz. I-IV, of the book, which are taken from the course of the Winter Semester of 1861—62, form an introduction to the theory of elliptic functions, the line of thought being the same in essentials as in the first edition of Briot and Bouquet's *Théorie des fonctions doublement périodiques.* As stated by Stahl in his preface, "the peculiarities of Riemann's treatment lie *first* in the abundant use of geometrical presentations, which bring out in a plastic manner the essential properties of the elliptic functions and at the same time immediately throw light on the fundamental values and the reality relations of the functions and integrals which are also particularly important for the applications; and *secondly* in the synthetic treatment of analytic problems which builds up the expression for the functions and integrals alone on the ground of their characteristic properties and almost without calculation from the given element and thereby guarantees an all-sided look into the nature of the problem and the variety of its solutions. Because of these properties, Riemann's course of lectures forms an important complement to the analytical mode of treatment as is, in connection with Weierstrass's theory, almost exclusively cultivated in recent times."

The importance of Riemann's work as a source of inspiration for successive generations of mathematicians right up to to-day has already been mentioned; as a proof of this importance may be given the following list of books which are mainly based on Riemann's ideas :—

F. Klein's *Riemanns Theorie der algebraischen Funktionen und ihrer Integrale*, Leipsic, 1882.

F. Klein's *Riemannsche Flächen*, Göttingen, 1894.

F. Prym's *Untersuchungen über die Riemannsche Thetaformel und die Riemannsche Charakteristikentheorie*, Leipsic, 1882.

A. Krazer's *Theorie der zweifach unendlichen Thetareihen auf Grund der Riemannschen Thetaformel*, Leipsic, 1882,

G. Rost's *Theorie der Riemannschen Thetafunction*, 1902.

C. Neumann's Riemann's *Theorie der Abelschen Integrale*, Leipsic, 1865.

F. Lindemann's *Untersuchungen über den Riemann-Rochschen Satz*, Leipsic, 1879.

H. Weyl's *Die Idee der Riemannschen Fläche*, Leipsic, 1923.

The following glowing tribute to Riemann's genius as the man who introduced what are called Riemann's surfaces into Analysis may be quoted here :—"Here and there one comes across the conception as if the *Riemann's surface* were nothing more than an image, than a means (it is admitted, *very* valuable, *very* suggestive) for bringing to one's mind, and giving a clear idea of; the multiple-valuedness of functions. This conception is from the bottom a perverse one. The Riemann's surface is an indispensable and real part of the theory, it is straightways its foundation ; it is also not something which is *a posteriori* more or less artificially distilled out of the analytical functions, but ought to be treated as the rich soil on which the functions can first of all grow and thrive. It is to be frankly admitted that Riemann himself has somewhat glossed over the true relation of functions to Riemann's surface by the form of his exposition—perhaps, only because he did not wish to approach his contemporaries with altogether foreign ideas. ... And indeed there is no doubt possible on this, that first in Klein's conception the fundamental thoughts of Riemann become fully current in their natural simplicity, their living and effective force".

The followiug lines, in which Riemann and Weierstrass are compared, will interest the reader :—

"Riemann and Weierstrass are without doubt the two greatest mathematicians of the second half of the 19th century. However zealously our science may be advanced, however numerous are those who have won real credit as investigators and teachers in connection with its progress, with these two no other can be matched; these two stand as truly great, perhaps equally great. Both devoted themselves essentially to Pure Analysis. Also both recognized that the representation of a function by means of a formula, as the older Mathematics did, is suited to disclose its properties only in the simplest cases, and that the characterization of a function leads to the goal in all cases. Accordingly, both required new methods in order to investigate the properties of a function in a way unknown until then. First these methods had to be created by both and in them they offered to their contemporaries and posterity an instrument which is suited to advance powerfully the most different branches of Mathematics. But the method is fundamentally different with each.

"Riemann starts from the geometrical representation. He represents the complex values of the variable as well as those of the function by all the points of the plane and considers the relation which exists after such representation between the two planes. Thereby the further progress is rendered very easy for Riemann whilst he can make the geometrical perception useful for Analysis. Also he succeeds in forcibly pushing back the ugliest and the most disagreeable part of the mathematical investigation, viz. the distressing working out of formulae, and in substituting pure thought-work for such part.

"Weierstrass approaches the representation by means of formulae; however, he does not calculate directly by means of them but connects only his deliberations with them. The calculations can then be, surely, not avoided to the same extent as in the case of Riemann, but instead the proofs require a rigour which up to that time had never been arrived at in Mathematics. Whilst the proper foundation of Riemann's theory could be first shown, very late and under the direct influence of Weierstrass's view; the structure which Weierstrass has erected is firmly established in its fundamentals and all the parts fit one another well. To mention two classical examples, let us compare the exactions by means of which Weierstrass introduces the elliptic and Abelian functions with the properties from which Riemann starts for the Gaussian P-functions: then one shall be inclined to give to the Weierstrassian method preeminence, even if the ease with which Riemann's method works guarantees to it an obvious advantage. However, I need not go further in these abstract statements. Since I have mentioned the name of Riemann, I must at least add that these two prominent mathematicians gave to each other a mutual recognition which passed into true friendship and that specially Weierstrass could never feel satisfied by doing all he could to place in true light the deserts of his rival and that he mourned Riemann's premature death in a most heartfelt manner"*

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* Killing, l. c., pp. 17–18.

Klein says:* "Riemann is the man of brilliant intuition. By his comprehensive genius he excels all his contemporaries. Where his interest is roused, he begins afresh without allowing himself to be misled by tradition and without recognizing the necessity of a system.

"Weierstrass is in the first rank of logicians; he proceeds slowly, systematically, step by step. Where he works, he aims at finality.

"With reference to the outer activity, it may be noted: After quiet preparation Riemann broke out like a bright meteor to fade away again soon; he limited his activity only to 15 years; 1851 is the year of his dissertation, 1862 of his illness, 1866 of his death.

"Weierstrass could work himself out slowly. He begins already in 1843 with some remarks on the analytical faculties; in 1897 he dies advanced in age and after an opulent life."

On the influence of Riemann on modern Mathematics, Klein makes the following interesting observations†: "It was obligatory on me to state so much in order that one may recognize to some extent already how the matchless genius of Riemann went ahead of his times and so influenced in a wide-reaching manner the production of the far future. Certainly the keystone in the building up of every mathematical theory is to bring the convincing proof for every assertion. Certainly Mathematics passes verdict against itself if it waives the convincing proof. The secret of ingenious productivity will eternally rest on this, to frame new questions, to divine new theorems, to deduce the valuable results and connections. Without the creation of new points of view, without the laying down of new goals, Mathematics would, in the rigour of its logical demonstration, soon exhaust itself and begin to stagnate, whilst it will fall short of stuff (to work on). So has Mathematics, in a certain sense, been advanced most by those who distinguished themselves more by intuition than by rigorous demonstration. There is no doubt that Riemann is that mathematician of the last decades whose posthumous influence is the greatest to-day.

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* Klein's *Entwicklung*, l. c., p. 24[illegible].

† l. c., pp. 271—272.

"Riemann's ideas, which had to influence in such a fundamental manner the devolopment of our modern theory of functions, have spread only slowly and quite gradually. His publications did not, as one would like to believe to-day, work as revelations and cause a sudden revolution in the mathematical views of his times. That may be due to the fact that Riemann's own publications were at first very difficult to understand, surely because of their terseness and the introduction of new and quite extraordinary notions."

Professor Castelnuovo also has given expression to a similar opinion about the influence of Riemann. In his lecture at the Bologna Congress on algebraic geometry, Professor Castelnuovo said: "It is known how this theory which carried a new spark of life into the field of algebraic geometry is due to one of the great mathematical genii of the last century, Bernardo Riemann, whose gigantic figure appeared more and more imposing the remoter one was from the times in which he lived."*

Atti, t, 1, pp. 192—193.

LIST OF RIEMANN'S LECTURES.*

Year and Semester	Subject of the Lectures	Number of hours a week	Remarks
1854—55 Winter	The theory of the integration of the partial differential equations together with the applications of the same to different problems of Physics	4	
1855 Summer	Definite integrals	4	
1855-56 Winter	The functions of a complex variable, in particular elliptic and Abelian	3	
1856 Summer	The mathematical theory of the elasticity of rigid bodies	4	
	Selected physical problems	2	
1856-57 Winter	The functions of a complex variable, in particular hypergeometric series and allied transcendentals	3	
1857 Summer	The theory of elliptic and Abelian functions	5	

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* See the *Gött. Nachrichten* for 1854—1866 ; also *Nachträge z. Riemann's Math Werke,* 1902, pp. 114—115.

Year and Semester	Subject of the Lectures	Number of hours a week	Remarks
1857-58 Winter	The theory of elliptic and Abelian functions	4	
1858 Summer	The mathematical theory of electricity and of magnetism	4	
	Selected physical problems	1	Public
1859-59 Winter	Functions of a variable, in particular hypergeometric series and allied transcendentals	4	
	Higher Mechanics	4	
1859 Summer	The mathematical theory of gravitation, of magnetism and of electricity	4	Mondays, Tuesdays Thursdays, Fridays at 9 A. M.
	Elliptic functions	4	Same days at 5 P. M.
1859-60 Winter	The mathematical theory of the elasticity of rigid bodies	4	
1860 Summer	The mathematical theory of gravitation, of electricity and of magnetism	4	
	The method of least squares	2	Public

Year and Semester	Subject of the Lectures	Number of hours a week	Remarks
1860—61 Winter	The theory of the partial differential equations with applications to physical problems	5	
1861 Summer	The mathematical theory of gravitation, of electricity and of magnetism	5	
	Functions of a complex variable, in particular elliptic and Abelian	4	
1861—62 Winter	Continuation of the lectures on the theory of the functions of a complex variable	3	Gratis
1862 Summer	The theory of the partial differential equations with applications to physical questions	5	
1862—63 Winter	The mathematical theory of gravitation, of magnetism and of electricity	5	
1863 Summer	As in Summer, 1862		
1863—64 Winter	Selected chapters of mathematical Physics		
1864 Summer	As in Summer, 1862		

ERRATA.

Page 18, Last line of the 2nd foot-note : *For* "n Gauss's" *read* "in Gauss's".

Page 21, Line 3 from bottom : *For* "faithfull" *read* "faithful."

Page 22, Line 5 of the footnote : *For* "both are" *read* "both p and q are"

Page 23, Line 4 from bottom : *For* "c 1" *read* "$c-1$".

Page 23, Last line : *For* "pp. 379" *read* "pp. 120—379"

Page 28, Line 4 ; *For* "substantiatly" *read* "substantially."

Page 38, Line 8 : *For* „Desquisitiones" *read* "Disquisitiones."

Page 38, Line 7 from bottom : *For* "182" *read* "1822"

Page 52, Line 1 : *For* "XVII" *read* "XVIII."

Page 63, Last line : *For* "be" *read* "he".

Page 81, Line 6 from bottom : *For* " i sues" *read* "issues".

Page 85, Lines 10 and 11 : Omit"; a list of such publications is given at the end of this chapter".

Page 97, Line 6 : *For* "of of" *read* "of".

Page 105, Line 18 : *For* "peneteration" *read* "penetration".

Page 107, Line 1 : *For* "M," *read* "M."

... ... Line 3 : *For* "mathematicions" *read* "mathematicians".

Page 109, Line 4 from bottom of the footnote : *For* "ruthlesly destroyed," *read* "ruthlessly destroyed." Line 3 from bottom of the footnote : *For* "wall," *read* "wall :"

Page 111, Line 6 from bottom : *For* "contry" *read* "century."

Page 123, Line 6 from bottom : *For* "pp. (" *read* "pp".

Page 124, Line 2 from bottom : *For* "staid" *read* "stayed".

Page 138, Line 2 from bottom : *For* "followign" *read* "following".

Page 144, Line 13 : *For* "whatsoover" *read* "whatsoever".

Page 161, Line 11 : *For* "Leavig" *read* "Leaving".

Page 183, Line 21 : Omit the cross †.

Page 184, Line 2 : *For* "esults" *read* "results",

Page 190, Line 15 : *For* "$A^{(m-2}$" *read* "$A^{(m-2)}$".

Page 190, Line 17 : *For* "emmerated" *read* "enumerated".

Page 207, Line 21 : *For* "and the" *read* "and".

Page 214, Line 8 : *For* "were be" *read* "were to be".

Page 222, Line 15 : *For* "Piücker" *read* "Plücker"

Page 223, Line 23 : *For* "recieved" *read* "received".

Page 225, Line 2 : *For* "differental" *read* "differential".

Page 228, Line 10 : *For* "s exactly" *read* "is exactly".

Page 231, Line 8 : *For* "expotential" *read* "exponential".

Page 237, Line 2 : *For* "*Dyz*" *read* "2 *Dyz*".

Page 252, Line 6 from bottom : *For* "*w*" *read* "ω"

Page 260 Line 14 : *For* "hood" *read* "bourhood."

Page 260, Line 4 from bottom : *For* “*i. e,*” *read* “*i. e.,*”

Page 262, Line 12 from bottom : *For* “function” *read* “functions”

Page 262, Line 5 from bottom : *For* “Eigeinschaften” *read* “Eigenschaften”

Page 263, Line 6 : *For* “funotion” *read* “function.”

Page 275, Line 3 from bottom : *For* $\frac{``dx}{\surd R(x)}$. *read* $\frac{``dx"}{\surd R(x)}$,

Page 279, Line 9 : *For* “geodesie” *read* “geodesic.”

Page 282, *For* “1857” ummer *read* “1857” Summer

Page 287, Fourth Row : *For* “Abelian “funtions” *read* “Abelian functions”

Page 289, First Row : *For* “Lecturss” *read* “Lectures.”

Page 295, Lines 8 and 9 : *For* “perception (Anschaung)” *read* “intuition (Anschauung).”

Page 296, Line 8 : *For* “equations,” *read* “equations.”

Page 300, Line 3 : *For* “occuring” *read* “occurring.”

age 300, Line 5 : *For* concieve” *read* conceive.”

Page 300, 2nd footnote : *For* “517” *read* “549.”

Page 300, 3rd footnote : *For* “*Werke*, pp. 45—46” *read* “*Werke*, 1892, pp. 44—45”.

Page 303. Omit the footnote.

Page 305, Line 2 from bottom : *For* “functlons” *read* “functions”.

Page 309, Line 2 : *For* “veen” *read* “been”

Page 309, Line 15 : *For* “Riemann” *read* ““Riemann”

Page 325, Footnote : *For* “ .c,” *read* ‘ l. c.”

BY GANESH PRASAD.

(1) **Constitution of Matter and analytical theories** of heat, (*Memoirs of the Royal Society of Sciences of Göttingen*, 1903), Weidmannsche Buchhandlung, Berlin. .

(2) **A text-book of Differential Calculus,** Longmans, Green and Co., London, 1910. (Revised edition, 1931).

(3) **A text-book of Integral Calculus, Longmans,** Green and Co., London, 1910. Second edition, 1929).

(4) **Mathematical Research in the last 20 years** (Presidential Address delivered to the Benares Mathematical Society in 1920), Walter de Gruyter and Co., Berlin.

(5) **German Translation of the preceding by** Dr. F. Lange, Walter de Gruyter and Co, Berlin.

(6) **Partial Differential Equations of Mathematical** Physics (Readership Lectures delivered at the Patna University in 1921), Patna University.

(7) **An Introduction to elliptic functions and** higher transcendentals, 1928, Calcutta University Press.

(8) **Six lectures on recent researches in the** theory of Fourier series, 1928, Calcutta University Press.

(9) **A treatise on Spherical Harmonics and the** functions of Bessel and Lamé, Part I, 1930, Benares Mathematical Society, Benares.

(10) **Six lectures on recent researches about the** mean-value theorem of Differential Calculus, 1931, Calcutta University Press.

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(11) **A treatise on Spherical Harmonics and** the functions of Bessel and Lamé, Part II, 1932, Benares Mathematical Society, Benares.

(12) **Some great mathematicians of the 19th** century : their lives and their works, Vol. II (To appear in October, 1933).

(13) **An Introduction to Integral Equations** (To appear in 1933).